7.2: Spherical Symmetry

A little more work is required if we want to link the existence of Killing vectors to the existence of a specific symmetry such as spherical symmetry. When we talk about spherical symmetry in the context of Newtonian gravity or Maxwell’s equations, we may say, “The fields only depend on \( r \),” implicitly assuming that there is an \( r \) coordinate that has a definite meaning for a given choice of origin. But coordinates in relativity are not guaranteed to have any particular physical interpretation such as distance from a particular origin. The origin may not even exist as part of the spacetime, as in the Schwarzschild metric, which has a singularity at the center. Another possibility is that the origin may not be unique, as on a Euclidean two-sphere like the earth’s surface, where a circle centered on the north pole is also a circle centered on the south pole; this can also occur in certain cosmological spacetimes that describe a universe that wraps around on itself spatially.

We therefore define spherical symmetry as follows. A spacetime \( (S) \) is spherically symmetric if we can write it as a union \( (S = \bigcup s_{r,t}) \) of nonintersecting subsets \( s_{r,t} \), where each \( s \) has the structure of a two-sphere, and the real numbers \( r \) and \( t \) have no preassigned physical interpretation, but \( s_{r,t} \) is required to vary smoothly as a function of them. By “has the structure of a two-sphere,” we mean that no intrinsic measurement on \( (s) \) will produce any result different from the result we would have obtained on some two-sphere. A two-sphere has only two intrinsic properties:

1. it is spacelike, i.e., locally its geometry is approximately that of the Euclidean plane;
2. it has a constant positive curvature.

If we like, we can require that the parameter \( r \) be the corresponding radius of curvature, in which case \( t \) is some timelike coordinate.

To link this definition to Killing vectors, we note that condition 2 is equivalent to the following alternative condition: (2’) The set \( s \) should have three Killing vectors (which by condition 1 are both spacelike), and it should be possible to choose these Killing vectors such that algebraically they act the same as the ones constructed explicitly in example 4 in section 7.1. As an
example of such an algebraic property, Figure \(\PageIndex{1}\) shows that rotations are noncommutative.

Figure \(\PageIndex{1}\): Performing the rotations in one order gives one result, 3, while reversing the order gives a different result, 5.

Example 7: A cylinder is not a sphere

- Show that a cylinder does not have the structure of a two-sphere.
- The cylinder passes condition 1. It fails condition 2 because its Gaussian curvature is zero. Alternatively, it fails condition 2' because it has only two independent Killing vectors (example 3).

Example 8: A plane is not a sphere

- Show that the Euclidean plane does not have the structure of a two-sphere.
- Condition 2 is violated because the Gaussian curvature is zero. Or if we wish, the plane violates 2' because \(\partial_x\) and \(\partial_y\) commute, but none of the Killing vectors of a 2-sphere commute.

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