14.7: Dissipative Lagrangians

Energy dissipation is an irreversible process that plays an important role for most physical systems encountered in nature. This irreversibility contrasts with the reversible nature of the basic models employed to describe conservative systems. Dissipation for an observed system usually arises from interactions between the observed system and a bath of unobserved systems that absorb the energy. Usually the detailed structure of the many systems that absorb the dissipated energy is irrelevant for the calculation of the dissipation. However, calculation of the interactions, and the transition from reversibility to irreversibility, are challenging problems to solve. In the Newtonian formulation the dissipation can be handled via a phenomenological approach. Unfortunately incorporating dissipative processes into the Lagrangian and Hamiltonian variational framework is more difficult. This difficulty stems from the fact that these variational formulations were derived from d’Alembert’s principle which assumes that the virtual work done by the constraint forces is zero, which is not true for dissipative forces.

As discussed in chapter 8?7, the following three approaches can be used to introduce dissipative forces into Lagrangian mechanics.

1. The dissipative force can be introduced as an external generalized force.

2. For the special case of linear dissipation, it is possible to use the Rayleigh dissipation function

\[ F = 2 \sum \dot{\gamma} \xi (13.33) \]
as discussed in chapter 8?7?2. Note that

\[ 2F = \]

\[ \frac{\partial L}{\partial q \dot{q}} \]

(13.34)

which is the rate of energy loss due to the dissipative forces involved.

3. Extensions of Lagrangian mechanics using non-standard Lagrangians can be used that build dissipation directly into the Lagrangian. This can allow exploitation of Lagrangian mechanics for a wide range of dissipative systems.

The use of non-standard Lagrangians is based on the inverse variational problem where known second-order equations of motion, plus the inverse variational approach, are used to derive a Lagrangian or Hamiltonian that generates the assumed equations of motion. Non-standard Lagrangians can have very different functional dependences on \( q', q \) and \( q \) compared with standard Lagrangians, and yet still can lead to the required equations of motion, the generalized momenta, and the corresponding Hamiltonian, needed to solve problems in classical mechanics. The reason for exploring the capabilities of use of non-standard Lagrangians is that they have the potential to eliminate some of the limitations endemic to Lagrangian and Hamiltonian mechanics.

Dissipation plays a prominent role in the burgeoning field of non-linear dynamical systems in classical mechanics. This prominence has stimulated recent studies of the applicability of standard, and non-standard, Lagrangians to a wide range of dissipative dynamical systems. Musielak et al, and others, [Mus08a, Mus08b, Cei10] considered dynamical systems that were described by equations of motion with first-order time-derivative dissipative terms of even and odd powers, and coefficients varying in time or space. They found that there are at least three different classes of equations of motion, two of which use standard Lagrangians and can be classified as general. However, the third class is special in that it can be derived only using non-standard Lagrangians. Each general class has a subset of equations with non-standard Lagrangians. The existence of standard Lagrangians is limited to equations of motion with either time-dependent coefficients plus linear dissipative terms, or space-dependent coefficients and quadratic dissipative terms. However, the equations of motion that can be derived from non-standard Lagrangians are restricted by conditions that must be satisfied by the coefficients and functions of these equations. Although these non-standard Lagrangians may have restricted applicability, they do provide hope that such techniques can be used to broaden the scope of problems that can be addressed using the basic Lagrangian and Hamiltonian mechanics formalisms. Note that, even though Lagrange published his treatise on analytical mechanics in 1788, fundamental problems remain to be solved in order to attain the full potential capabilities of analytical mechanics.