10.1: Lorentz Transformations of Energy and Momentum

As you may know, like we can combine position and time in one four-vector \(x=(\vec{x}, ct)\), we can also combine energy and momentum in a single four-vector, \(p=(\vec{p}, E/c)\). From the Lorentz transformation property of time and position, for a change of velocity along the \(x\)-axis from a coordinate system at rest to one that is moving with velocity \(v_x,0,0,0\) we have

\[
\begin{align}
x' &= \gamma(v) (x-v/c t), \\
t' &= \gamma (t-xv x/c^2),
\end{align}
\]

we can derive that energy and momentum behave in the same way,

\[
\begin{align}
p'_x &= \gamma(v) (p_x - E v/c^2) \\
E' &= \gamma(|u|) m_0 c^2.
\end{align}
\]

To understand the context of these equations remember the definition of \(\gamma\)

\[
\gamma(v) = \frac{1}{\sqrt{1-\beta^2}}
\]

and

\[
\beta = \frac{v}{c}.
\]

In Equation \ref{eqLorentzE}, we have also re-expressed the momentum energy in terms of a velocity \(\vec{u}\), measured relative to the rest system of a particle, the system where the three-momentum \(\vec{p}=0\). This is now all these exercises would be interesting mathematics but rather futile if there was no further information. We know however that the full four-momentum is conserved, i.e., if we have two particles coming into a collision and two coming out,
the sum of four-momenta before and after is equal,

\[
\begin{aligned}
E^{\text{in}}_1 + E^{\text{in}}_2 &= E^{\text{out}}_1 + E^{\text{out}}_2, \\
\vec{p}^{\text{in}}_1 + \vec{p}^{\text{in}}_2 &= \vec{p}^{\text{out}}_1 + \vec{p}^{\text{out}}_2.
\end{aligned}
\]