3.2: Normalization of the Wavefunction

Now, a probability is a real number lying between 0 and 1. An outcome of a measurement that has a probability 0 is an impossible outcome, whereas an outcome that has a probability 1 is a certain outcome. According to Equation (3.2), the probability of a measurement of $x$ yielding a result lying between $-\infty$ and $+\infty$ is

$$P_{x\in\, -\infty:\infty}(t) = \int_{-\infty}^{\infty} |\psi(x,t)|^2 \, dx.$$ 

However, a measurement of $x$ must yield a value lying between $-\infty$ and $+\infty$, because the particle has to be located somewhere. It follows that $P_{x\in\, -\infty:\infty}=1$, or

$$\int_{-\infty}^{\infty} |\psi(x,t)|^2 \, dx = 1,$$

which is generally known as the normalization condition for the wavefunction.

For example, suppose that we wish to normalize the wavefunction of a Gaussian wave-packet, centered on $(x=x_0)$, and of characteristic width $(\sigma)$ (see Section 2.9): that is, $\psi(x) = \psi_0 \exp\left(-\frac{(x-x_0)^2}{4\sigma^2}\right)$. In order to determine the normalization constant $\psi_0$, we simply substitute Equation (3.5) into Equation (3.4) to obtain $\int |\psi_0|^2 \, dx = 1$. Changing the variable of integration to $y=(x-x_0)/(\sqrt{2}\sigma)$, we get

$$\int |\psi_0|^2 \, dx = \sqrt{2}\sigma \int_{-\infty}^{\infty} e^{-y^2} \, dy = 1.$$ 

However,

$$\int_{-\infty}^{\infty} e^{-y^2} \, dy = \sqrt{\pi},$$

which implies that $|\psi_0|^2 = \frac{1}{(2\pi\sigma^2)^{1/2}}$. Hence, a general normalized Gaussian wavefunction takes the form

$$\psi(x) = \frac{e^{i\varphi}}{(2\pi\sigma^2)^{1/4}} \exp\left(-\frac{(x-x_0)^2}{4\sigma^2}\right),$$

where $(\varphi)$ is an arbitrary real phase-angle.

It is important to demonstrate that if a wavefunction is initially normalized then it stays normalized as it evolves in time according to Schrödinger’s equation. If this is not the case then the probability interpretation of the wavefunction is untenable, because it does not make sense for the probability that a measurement of $(x)$ yields any possible outcome (which is, manifestly, unity) to change in time. Hence, we require that $\frac{d}{dt} \int_{-\infty}^{\infty} |\psi(x,t)|^2 \, dx = 0$ for
wavefunctions satisfying Schrödinger’s equation. The previous equation gives

\[
\frac{d}{dt}\int_{-\infty}^{\infty} \psi^{\ast}(x,t)\psi(x,t)\,dx = \int_{-\infty}^{\infty} \left( \frac{\partial}{\partial t}\psi^{\ast}(x,t)\psi(x,t) + \psi^{\ast}(x,t)\frac{\partial\psi(x,t)}{\partial t} \right)\,dx = 0. \]

Now, multiplying Schrödinger’s equation by \(\psi^{\ast}/(\text{i}\,\hbar)\), we obtain

\[
\psi^{\ast}(x,t)\frac{\partial \psi(x,t)}{\partial t} = \frac{\text{i}\,\hbar}{2\,m}\psi^{\ast}(x,t)\frac{\partial^2\psi(x,t)}{\partial x^2} - \frac{\text{i}}{\hbar}V|\psi(x,t)|^2.
\]

The complex conjugate of this expression yields

\[
\psi(x,t)\frac{\partial \psi^{\ast}(x,t)}{\partial t} = -\frac{\text{i}\,\hbar}{2\,m}\psi(x,t)\frac{\partial^2\psi^{\ast}(x,t)}{\partial x^2} + \frac{i}{\hbar}V|\psi(x,t)|^2.
\]

[because \((A\,B)^{\ast} = A^{\ast}\,B^\ast\), \((A^\ast)^{\ast} = A\), and \(\{(\text{rm}\,i)^\ast\}^{\ast} = -\{(\text{rm}\,i)\}\).]

Summing the previous two equations, we get

\[
\frac{\partial \psi^{\ast}(x,t)}{\partial t} - \psi^{\ast}(x,t)\frac{\partial \psi(x,t)}{\partial t} = \frac{\text{i}\,\hbar}{2\,m}\left( \psi^{\ast}(x,t)\frac{\partial^2\psi(x,t)}{\partial x^2} - \psi(x,t)\frac{\partial^2\psi^{\ast}(x,t)}{\partial t^2} \right) = \frac{\text{i}\,\hbar}{2\,m}\frac{\partial}{\partial x}\left( \psi^{\ast}(x,t)\psi(x,t) - \psi(x,t)^\ast\psi^{\ast}(x,t) \right) \bigg|_{-\infty}^{\infty}.
\]

Equations ((e3.12)) and ((e3.15)) can be combined to produce \(\frac{d}{dt}\int_{a}^{b} |\psi(x,t)|^2\,dx = \frac{\text{i}\,\hbar}{2\,m}\left( \psi^{\ast}(x,t)\psi(x,t) - \psi(x,t)^\ast\psi^{\ast}(x,t) \right)\bigg|_{-\infty}^{\infty} = 0.\) The previous equation is satisfied provided \(\{(\text{rm}\,i)^\ast\}^{\ast} = \{(\text{rm}\,i)\}\) converges. Hence, we conclude that all wavefunctions that are square-integrable [i.e., satisfy the integral condition ((e3.4))] have the property that if the normalization condition ((e3.4)) is satisfied at one instant in time then it is satisfied at all subsequent times.

It is also possible to demonstrate, via very similar analysis to that just described, that

\[
\frac{d}{dt}\int P_\ast(x)^\ast \,\,dx = 0, \quad \text{where } P_\ast(x) \text{ is defined in Equation ((e3.2))}, \quad \text{and}
\]

\[
\int_{a}^{b} |\psi(x,t)|^2\,dx \propto \int_{a}^{b} |\psi(x,t)|^2\,dx. \quad \text{In the following, all wavefunctions are assumed to be square-integrable and normalized,}
\]

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