## 10.2: Angular Momentum in Hydrogen Atom

In a hydrogen atom, the wavefunction of an electron in a simultaneous eigenstate of \( (L^2) \) and \( (L_z) \) has an angular dependence specified by the spherical harmonic \( Y_{l,m}(\theta,\phi) \). (See Section [sharm].) If the electron is also in an eigenstate of \( (S^2) \) and \( (S_z) \) then the quantum numbers \( \langle s \rangle \) and \( \langle m_s \rangle \) take the values \( \langle 1/2 \rangle \) and \( \langle \pm 1/2 \rangle \), respectively, and the internal state of the electron is specified by the spinors \( \chi_{\pm} \). (See Section [spauli].) Hence, the simultaneous eigenstates of \( (L^2) \), \( (S^2) \), \( (L_z) \), and \( (S_z) \) can be written in the separable form \( \psi^{(1)}_{l,1/2;m,\pm 1/2} = Y_{l,m}\chi_{\pm} \) Here, it is understood that orbital angular momentum operators act on the spherical harmonic functions, \( Y_{l,m} \), whereas spin angular momentum operators act on the spinors, \( \chi_{\pm} \).

Because the eigenstates \( \langle \psi^{(1)}_{l,1/2;m,\pm 1/2} \rangle \) are (presumably) orthonormal, and form a complete set, we can express the eigenstates \( \langle \psi^{(2)}_{l,1/2;j,m_j} \rangle \) as linear combinations of them. For instance,

\[
\psi^{(2)}_{l,1/2;j,m+1/2} = \alpha \psi^{(1)}_{l,1/2;m,1/2} + \beta \psi^{(1)}_{l,1/2;m+1,-1/2},
\]

where the number of \( \psi^{(1)} \) states that can appear on the right-hand side of the previous expression is limited to two by the constraint that \( m_j = m + m_s \) [see Equation ([e11.23])], and the fact that \( m_s \) can only take the values \( \pm 1/2 \). Assuming that the \( \psi^{(2)} \) eigenstates are properly normalized, we have

\[
\alpha^2 + \beta^2 = 1.\]

Now, it follows from Equation ([e11.26]) that

\[
J^2 \psi^{(2)}_{l,1/2;j,m+1/2} = j(j+1)\hbar^2 \psi^{(2)}_{l,1/2;j,m+1/2},
\]

where [see Equation ([e11.12])] \( J^2 = L^2 + S^2 + 2L_zS_z + L_+S_- + L_-S_+ \) Moreover, according to Equations ([e11.28]) and ([e11.29]), we can write \( \psi^{(2)}_{l,1/2;j,m+1/2} = \alpha Y_{l,m}\chi_+ + \beta, \)
Recall, from Equations (e11.32) and (e11.34), that
\[ L_+ Y_{l,m} = [l(l+1)-m(m+1)]^{1/2} \hbar Y_{l,m+1}, \]
\[ L_- Y_{l,m} = [l(l+1)-m(m-1)]^{1/2} \hbar Y_{l,m-1}. \]
By analogy, when the spin raising and lowering operators, \( S_\pm \), act on a general spinor, \( \chi_{s,m_s} \), we obtain
\[ S_+ \chi_{s,m_s} = [s(s+1)-m_s(m_s+1)]^{1/2} \hbar \chi_{s,m_s+1}, \]
\[ S_- \chi_{s,m_s} = [s(s+1)-m_s(m_s-1)]^{1/2} \hbar \chi_{s,m_s-1}. \]
For the special case of spin one-half spinors (i.e., \( s=1/2, m_s=\pm 1/2 \)), the previous expressions reduce to
\[ S_+ \chi_+ = S_- \chi_- = 0, \]
\[ S_\pm \chi_\mp = \hbar \chi_\mp. \]
It follows from Equations (e11.32) and (e11.34)–(e11.39) that
\begin{align*}
J^2 Y_{l,m} \chi_+ &= [l(l+1)+3/4+m] \hbar^2 Y_{l,m} \chi_+ + [l(l+1)-m(m+1)]^{1/2} \hbar^2 Y_{l,m+1} \chi_- , \\
J^2 Y_{l,m+1} \chi_- &= [l(l+1)+3/4-m-1] \hbar^2 Y_{l,m+1} \chi_- + [l(l+1)-m(m+1)]^{1/2} \hbar^2 Y_{l,m} \chi_+ .
\end{align*}
Hence, Equations (e11.31) and (e11.33) can be solved to give
\[ x(x+1) = l(l+1), \]
\[ \frac{\alpha}{\beta} = \frac{[(l-m)(l+m+1)]^{1/2}}{x-m}. \]
It follows that \( x=l \) or \( x=-l-1 \), which corresponds to \( j=1/2 \) or \( j=-1/2 \), respectively. Once \( x \) is specified, Equations (e11.30) and (e11.45) can be solved for \( \alpha \) and \( \beta \). We obtain
\[ \psi^{(2)}_{l+1/2,m+1/2} = \left( \frac{l+m+1}{2l+1} \right)^{1/2} \psi^{(1)}_{m,1/2} + \left( \frac{l-m}{2l+1} \right)^{1/2} \psi^{(1)}_{m+1,-1/2}, \]
\[ \psi^{(2)}_{l-1/2,m+1/2} = \left( \frac{l-m}{2l+1} \right)^{1/2} \psi^{(1)}_{m,1/2} - \left( \frac{l+m+1}{2l+1} \right)^{1/2} \psi^{(1)}_{m+1,-1/2}. \]
Here, we have neglected the common subscripts \( l, m \), et cetera. The information contained in Equations (e11.47)–(e11.50) is neatly summarized in Table 2. For instance, Equation (e11.47) is obtained by reading the first row of this table, whereas Equation (e11.50) is obtained by reading the second column. The coefficients in this type of table are generally known as Clebsch-Gordon coefficients.

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
\( (m, m+1/2) \) & \( (m+1, -1/2) \) & \( (m, m_{s1}) \) \\
\hline
\( [0.5ex] (l+1/2, m+1/2) \) & \( \left( \sqrt{(l+m+1)/(2l+1)} \right) \) & \( \left( \sqrt{(l-m)/(2l+1)} \right) \) \\
\hline
\end{tabular}
\caption{Clebsch-Gordon coefficients for adding spin one-half to spin \( l \).}
\end{table}
As an example, let us consider the \( \ell = 1 \) states of a hydrogen atom. The eigenstates of \( (L^2), (S^2), (L_z), \) and \( (S_z) \), are denoted \( \langle \psi^{(1)}_{m,m_s} \rangle \). Because \( (m) \) can take the values \( (-1,0,1) \), whereas \( (m_s) \) can take the values \( (\pm 1/2) \), there are clearly six such states: that is, \( \langle \psi^{(1)}_{1,\pm 1/2} \rangle, \langle \psi^{(1)}_{0,\pm 1/2} \rangle, \) and \( \langle \psi^{(1)}_{-1,\pm 1/2} \rangle \). Because \( (l=1) \) and \( (s=1/2) \) can be combined together to form either \( (j=3/2) \) or \( (j=1/2) \) (see previously), there are also six such states: that is, \( \langle \psi^{(2)}_{3/2,\pm 3/2} \rangle, \langle \psi^{(2)}_{3/2,\pm 1/2} \rangle, \) and \( \langle \psi^{(2)}_{1/2,\pm 1/2} \rangle \). According to Table \([12]\), the various different eigenstates are interrelated as follows:

\[
\begin{aligned}
\psi^{(2)}_{3/2,\pm 3/2} &= \psi^{(1)}_{\pm 1,\pm 1/2}, \\
\psi^{(2)}_{3/2,1/2} &= \frac{\sqrt{2}}{3} \psi^{(1)}_{0,1/2} + \frac{\sqrt{1}}{3} \psi^{(1)}_{1,-1/2}, \\
\psi^{(2)}_{1/2,1/2} &= \frac{\sqrt{1}}{3} \psi^{(1)}_{0,1/2} - \frac{\sqrt{2}}{3} \psi^{(1)}_{1,-1/2}, \\
\psi^{(2)}_{3/2,-1/2} &= \frac{\sqrt{1}}{3} \psi^{(1)}_{-1,1/2} + \frac{\sqrt{2}}{3} \psi^{(1)}_{0,-1/2},
\end{aligned}
\]

Thus, if we know that an electron in a hydrogen atom is in an \( \ell = 1 \) state characterized by \( (m=0) \) and \( (m_s=1/2) \), i.e., the state represented by \( \langle \psi^{(1)}_{0,1/2} \rangle \) then, according to Equation \((e11.59)\), a measurement of the total angular momentum will yield \( (j=3/2), (m=1/2) \) with probability \( (2/3) \), and \( (j=1/2), (m=1/2) \) with probability \( (1/3) \). Suppose that we make such a measurement, and obtain the result \( (j=3/2), (m=1/2) \). As a result of the measurement, the electron is thrown into the corresponding eigenstate, \( \langle \psi^{(2)}_{3/2,1/2} \rangle \). It thus follows from Equation \((e11.52)\) that a subsequent measurement of \( (L_z) \) and \( (S_z) \) will yield \( \langle m=0 \rangle, \langle m_s=1/2 \rangle \) with probability \( (2/3) \), and \( \langle m=1 \rangle, \langle m_s=-1/2 \rangle \) with probability \( (1/3) \).

Clebsch-Gordon coefficients for adding spin one-half to spin one. Only non-zero coefficients are shown.

\[
\begin{array}{cccccccc}
\langle -1, -1/2 \rangle & \langle -1, 1/2 \rangle & \langle 0, -1/2 \rangle & \langle 0, 1/2 \rangle & \langle 1, -1/2 \rangle & \langle 1, 1/2 \rangle & \langle m, m_s \rangle \\
\langle 3/2, -3/2 \rangle \\
\langle 3/2, -1/2 \rangle & \langle 3/2, 3/2 \rangle \\
\langle 1/2, -1/2 \rangle & \langle 1/2, 1/2 \rangle \\
\end{array}
\]
The information contained in Equations (1)–(4) is neatly summed up in Table [3]. Note that each row and column of this table has unit norm, and also that the different rows and different columns are mutually orthogonal. Of course, this is because the $\psi^{(1)}$ and $\psi^{(2)}$ eigenstates are orthonormal.

**Contributors and Attributions**

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