10.2: Angular Momentum in Hydrogen Atom

In a hydrogen atom, the wavefunction of an electron in a simultaneous eigenstate of \((L^2)\) and \((L_z)\) has an angular dependence specified by the spherical harmonic \((Y_{l,m}\theta,\phi))\). (See Section [sharm],) If the electron is also in an eigenstate of \((S^2)\) and \((S_z)\) then the quantum numbers \((s)\) and \((m_s)\) take the values \((1/2)\) and \((-1/2)\), respectively, and the internal state of the electron is specified by the spinors \((\chi_\pm)\). (See Section [spauli].) Hence, the simultaneous eigenstates of \((L^2)\), \((S^2)\), \((L_z)\), and \((S_z)\) can be written in the separable form \(\psi^{(1)}_{l,1/2;m,\pm 1/2} = Y_{l,m}\chi_\pm\). Here, it is understood that orbital angular momentum operators act on the spherical harmonic functions, \(Y_{l,m}\), whereas spin angular momentum operators act on the spinors, \(\chi_\pm\).

Because the eigenstates \(\psi^{(1)}_{l,1/2;m,\pm 1/2}\) are (presumably) orthonormal, and form a complete set, we can express the eigenstates \(\psi^{(2)}_{l,1/2;j,m+1/2}\) as linear combinations of them. For instance,

\[
\psi^{(2)}_{l,1/2;j,m+1/2} = \alpha Y_{l,m}\chi_+ + \beta Y_{l,m}\chi_- = \alpha Y_{l,m}\chi_+ + \beta Y_{l,m}\chi_-.
\]

Now, it follows from Equation ([e11.26]) that

\[
\alpha^2 + \beta^2 = 1,
\]

where \(\alpha\) and \(\beta\) are, as yet, unknown coefficients. Note that the number of \(\psi^{(1)}\) states that can appear on the right-hand side of the previous expression is limited to two by the constraint that \(m_j = m + m_s\) [see Equation ([e11.23])], and the fact that \(m_s\) can only take the values \((-1/2)\) or \((1/2)\). Assuming that the \(\psi^{(2)}\) eigenstates are properly normalized, we have

\[
\alpha^2 + \beta^2 = 1.
\]
Recall, from Equations \((e11.34)\) and \((e11.35)\), that

\[
L_+ Y_{l,m} = [l(l+1)-m(m+1)]^{1/2} \hbar Y_{l,m+1},
\]

\[
L_- Y_{l,m} = [l(l+1)-m(m-1)]^{1/2} \hbar Y_{l,m-1}.
\]

By analogy, when the spin raising and lowering operators, \((S_\pm)\), act on a general spinor, \((\chi_{s,m_s})\), we obtain

\[
S_+ \chi_{s,m_s} = [s(s+1)-m_s(m_s+1)]^{1/2} \hbar \chi_{s,m_s+1},
\]

\[
S_- \chi_{s,m_s} = [s(s+1)-m_s(m_s-1)]^{1/2} \hbar \chi_{s,m_s-1}.
\]

For the special case of spin one-half spinors (i.e., \((s=1/2, m_s=\pm 1/2)\)), the previous expressions reduce to

\[
S_+ \chi_{s} = S_- \chi_{s} = 0,
\]

\[
S_\pm \chi_{s} = \hbar \chi_{s}.
\]

It follows from Equations \((e11.32)\) and \((e11.34)-(e11.39)\) that

\[
(l-1/2, m+1/2) \quad \frac{\sqrt{(l+m+1)}}{2(l+1)} \quad \frac{\sqrt{(l-m)}}{2(l+1)}
\]

where \((l=j(j+1)-1(l+1)-3/4)\). Equations \((e11.42)\) and \((e11.43)\) can be solved to give

\[
x = \left\{ \begin{array}{ll}
(l-l) & \text{if } x < 0, \\
(l+l+1) & \text{if } x > 0.
\end{array} \right.
\]

The information contained in Equations \((e11.47)-(e11.50)\) is neatly summarized in Table \([t2]\). For instance, Equation \((e11.47)\) is obtained by reading the first row of this table, whereas Equation \((e11.50)\) is obtained by reading the second column. The coefficients in this type of table are generally known as *Clebsch-Gordon coefficients*. 

<table>
<thead>
<tr>
<th>(l)</th>
<th>(m)</th>
<th>(m+1)</th>
<th>(m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(l)</td>
<td>(m)</td>
<td>(m+1)</td>
<td>(m)</td>
</tr>
</tbody>
</table>

UC Davis ChemWiki is licensed under a Creative Commons Attribution-Noncommercial-Share Alike 3.0 United States License.
As an example, let us consider the \(l=1\) states of a hydrogen atom. The eigenstates of \(L^2\), \(S^2\), \(L_z\), and \(S_z\), are denoted \(|\psi^{(1)}_{m,m_s}\rangle\). Because \(m\) can take the values \(-1,0,1\), whereas \(m_s\) can take the values \(-1/2,1/2\), there are clearly six such states: that is, \(|\psi^{(1)}_{1,\pm 1/2}\rangle\), \(|\psi^{(1)}_{0,\pm 1/2}\rangle\), and \(|\psi^{(1)}_{-1,\pm 1/2}\rangle\). Because \(l=1\) and \(s=1/2\) can be combined together to form either \(j=3/2\) or \(j=1/2\) (see previously), there are also six such states: that is, \(|\psi^{(2)}_{3/2,\pm 3/2}\rangle\), \(|\psi^{(2)}_{3/2,\pm 1/2}\rangle\), and \(|\psi^{(2)}_{1/2,\pm 1/2}\rangle\). According to Table [12], the various different eigenstates are interrelated as follows:

\[
\begin{aligned}
\psi^{(2)}_{3/2,\pm 3/2} &= \psi^{(1)}_{\pm 1, \pm 1/2}, \\
\psi^{(2)}_{3/2,\pm 1/2} &= \sqrt{\frac{2}{3}}\psi^{(1)}_{0,1/2} + \sqrt{\frac{1}{3}}\psi^{(1)}_{1,-1/2}, \\
\psi^{(2)}_{1/2,\pm 1/2} &= \sqrt{\frac{1}{3}}\psi^{(1)}_{0,1/2} - \sqrt{\frac{2}{3}}\psi^{(1)}_{1,-1/2}, \\
\psi^{(2)}_{3/2,-1/2} &= \sqrt{\frac{1}{3}}\psi^{(1)}_{-1,1/2} + \sqrt{\frac{2}{3}}\psi^{(1)}_{0,-1/2}, \\
\end{aligned}
\]

Thus, if we know that an electron in a hydrogen atom is in an \(l=1\) state characterized by \(m=0\) and \(m_s=1/2\), i.e., the state represented by \(|\psi^{(1)}_{0,1/2}\rangle\) then, according to Equation ([e11.57]), a measurement of the total angular momentum will yield \(j=3/2\), \(m=0\) with probability \(2/3\), and \(j=1/2\), \(m=1/2\) with probability \(1/3\). Suppose that we make such a measurement, and obtain the result \(|\psi^{(2)}_{3/2,1/2}\rangle\). As a result of the measurement, the electron is thrown into the corresponding eigenstate, \(|\psi^{(2)}_{3/2,1/2}\rangle\). It thus follows from Equation ([e11.52]) that a subsequent measurement of \(L_z\) and \(S_z\) will yield \(|\psi^{(2)}_{3/2,1/2}\rangle\) with probability \(2/3\), and \(|\psi^{(2)}_{1/2,1/2}\rangle\) with probability \(1/3\).
The information contained in Equations ([ecgs])–([ecge]) is neatly summed up in Table [13]. Note that each row and column of this table has unit norm, and also that the different rows and different columns are mutually orthogonal. Of course, this is because the $\psi^{(1)}$ and $\psi^{(2)}$ eigenstates are orthonormal.

**Contributors and Attributions**

- **Richard Fitzpatrick** (Professor of Physics, The University of Texas at Austin)