12.4: Perturbation Expansion

Let us recall the analysis of Section 1.2. The \( \langle \psi_n \rangle \) are the stationary orthonormal eigenstates of the time-independent unperturbed Hamiltonian, \( H_0 \). Thus, \( \langle H_0 \rangle \psi_n = E_n \psi_n \), where the \( \langle E_n \rangle \) are the unperturbed energy levels, and \( \langle \psi_n \rangle = \delta_{nm} \). Now, in the presence of a small time-dependent perturbation to the Hamiltonian, \( H_1(t) \), the wavefunction of the system takes the form \( \psi(t) = \sum_n c_n(t) \exp(-\{i\omega_{nm}\}t)\psi_n \) where \( \omega_{nm} = (E_n - E_m)/\hbar \). The amplitudes \( \langle c_n(t) \rangle \) satisfy

\[
\frac{d c_n}{dt} = \sum_m H_{nm} \exp(\{i\omega_{nm}\}t) c_m,
\]

where \( H_{nm} = \langle n | H_1(t) | m \rangle \). Finally, the probability of finding the system in the \( n \)-th eigenstate at time \( t \) is simply \( P_n(t) = |c_n(t)|^2 \) (assuming that, initially, \( \sum_n |c_n|^2 = 1 \)).

Suppose that at \( t=0 \) the system is in some initial energy eigenstate labeled \( \langle i \rangle \). Equation (13.42) is, thus, subject to the initial condition \( c_n(0) = \delta_{ni} \). Let us attempt a perturbative solution of Equation (13.42) using the ratio of \( H_1 \) to \( H_0 \) (or \( H_{nm} / \hbar \omega_{nm} \), to be more exact) as our expansion parameter. Now, according to Equation (13.42), the \( c_n \) are constant in time in the absence of the perturbation. Hence, the zeroth-order solution is simply \( c_n^{(0)}(t) = \delta_{ni} \). The first-order solution is obtained, via iteration, by substituting the zeroth-order solution into the right-hand side of Equation (13.42). Thus, we obtain \( \frac{d c_n^{(1)}}{dt} = \sum_m H_{nm} \exp(\{i\omega_{nm}\}t) c_m^{(0)} = H_{ni} \exp(\{i\omega_{ni}\}t) \). The solution to the previous equation is \( c_n^{(1)}(t) = -\int_0^t H_{ni}(t') \exp(\{i\omega_{ni}\}t') dt' \). It follows that, up to first-order in our perturbation expansion,

\[
\langle c_n(t) \rangle = \delta_{ni} - \int_0^t H_{ni}(t') \exp(\{i\omega_{ni}\}t') dt'.
\]
different initial energy eigenstate labeled \( i \) at time \( t=0 \), is
\[
\langle P_{i\rightarrow f}(t) = |c_f(t)|^2 = \left| -\frac{i}{\hbar}\int_0^t H_{fi}(t')\exp(-i\omega_{fi}t')dt'\right|^2.\]
Note, finally, that our perturbative solution is clearly only valid provided
\[
|P_{i\rightarrow f}(t)| \ll 1.\]

Contributors and Attributions

- Richard Fitzpatrick (Professor of Physics, The University of Texas at Austin)