12.4: Perturbation Expansion

Let us recall the analysis of Section 1.2. The \(\langle \psi_n \rangle\) are the stationary orthonormal eigenstates of the time-independent unperturbed Hamiltonian, \(H_0\). Thus, \(\langle H_0 \rangle \langle \psi_n = E_n \rangle \langle \psi_n \rangle\), where the \(\langle E_n \rangle\) are the unperturbed energy levels, and \(\langle \psi_n | m \rangle = \delta_{nm}\). Now, in the presence of a small time-dependent perturbation to the Hamiltonian, \(H_1(t)\), the wavefunction of the system takes the form \(\psi(t) = \sum_n c_n(t) \exp(-i\omega_n t) \psi_n\) where \(\langle \psi_n | E_n = E_n \rangle \hbar \rangle\). The amplitudes \(\langle c_n(t) \rangle\) satisfy
\[i\hbar \frac{dc_n}{dt} = \sum_m H_{nm} \exp(i\omega_{nm} t) c_m\]
where \(H_{nm}(t) = \langle n | H_1(t) | m \rangle\) and \(\omega_{nm} = (E_n - E_m)/\hbar\). Finally, the probability of finding the system in the \(n\)th eigenstate at time \(t\) is simply \(P_n(t) = |c_n(t)|^2\) (assuming that, initially, \(\sum_n |c_n|^2 = 1\)).

Suppose that at \(t=0\) the system is in some initial energy eigenstate labeled \(i\). Equation \(\{e13.42\}\) is, thus, subject to the initial condition \(c_n(0) = \delta_{ni}\). Let us attempt a perturbative solution of Equation \(\{e13.42\}\) using the ratio of \(H_1\) to \(H_0\) (or \(H_{nm}\) to \(\hbar \omega_{nm}\), to be more exact) as our expansion parameter. Now, according to Equation \(\{e13.42\}\), the \(c_n\) are constant in time in the absence of the perturbation. Hence, the zeroth-order solution is simply \(c_n^{(0)} (t) = \delta_{ni}\). The first-order solution is obtained, via iteration, by substituting the zeroth-order solution into the right-hand side of Equation \(\{e13.42\}\). Thus, we obtain \(\{i\frac{\hbar}{\{}i\hbar\}\frac{dc_n}{dt} = \sum_m \exp(\{i \omega \{n m\}\}) \{c_m \}}\)
subject to the boundary condition \(c_n(0) = 0\). The solution to the previous equation is \(c_n^{(1)} (t) = i \frac{\hbar}{\{i\hbar\}} \int_0^t H_{ni}(t') \exp(\{i \omega \{n i\}\}) dt'\). It follows that, up to first-order in our perturbation expansion,
\[c_n(t) = \delta_{ni} - i \frac{\hbar}{\{i\hbar\}} \int_0^t H_{ni}(t') \exp(\{i \omega \{n i\}\}) dt'\] Hence, the probability of finding the system in some final energy eigenstate labeled \(f\) at time \(t\), given that it is definitely in a
different initial energy eigenstate labeled \( \langle i \rangle \) at time \( t=0 \), is \( \langle P_{\langle i \rangle \rightarrow \langle f \rangle} (t) = |c_f(t)|^2 = \left| \frac{-i}{\hbar} \int_0^t H_{fi}(t') \exp(-i \omega_{fi} t') dt' \right|^2 \rangle \). Note, finally, that our perturbative solution is clearly only valid provided \( |P_{\langle i \rangle \rightarrow \langle f \rangle} (t)| \ll 1 \).