14.2: Born Approximation

Equation ([e15.17]) is not particularly useful, as it stands, because the quantity \( f({\bf k}, {\bf k}') \) depends on the, as yet, unknown wavefunction \( \psi({\bf r}) \). [See Equation ([e5.12]).] Suppose, however, that the scattering is not particularly strong. In this case, it is reasonable to suppose that the total wavefunction, \( \psi({\bf r}) \), does not differ substantially from the incident wavefunction, \( \psi_0({\bf r}) \). Thus, we can obtain an expression for \( f({\bf k}, {\bf k}') \) by making the substitution \( \psi({\bf r}) \rightarrow \psi_0({\bf r}) = \sqrt{n}\exp(\,{\rm i}\, {\bf k}\cdot{\bf r}) \) in Equation ([e5.12]). This procedure is called the Born approximation.

The Born approximation yields

\[
[f({\bf k}, {\bf k}') \simeq \frac{m}{2\pi\hbar^2} \int \exp(\,{\rm i}\,(k-k')\cdot{\bf r}')V({\bf r}')\,d^3{\bf r}'.
\]

Thus, \( f({\bf k}, {\bf k}') \) becomes proportional to the Fourier transform of the scattering potential \( V({\bf r}) \) with respect to the wavevector \( \mathbf{q} = \mathbf{k} - \mathbf{k}' \).

For a spherically symmetric potential, \( f({\bf k}', {\bf k}) \simeq -\frac{2mV_0}{\hbar^2q^2+\mu^2} \int_0^\infty r'V(r')\sin(qr')\,dr' \) giving

\[f({\bf k}', {\bf k}) \simeq -\frac{2mV_0}{\hbar^2q^2+\mu^2} \int_0^\infty \sin(qr')\,dr'.
\]

Thus, in the Born approximation, the differential cross-section for scattering by a Yukawa potential is

\[
\frac{d\sigma}{d\Omega} \simeq \frac{4\pi m^2V_0^2}{\hbar^4\mu^2q^4}(q^2+\mu^2)^{-2}.
\]
The Yukawa potential reduces to the familiar Coulomb potential as \(\mu \rightarrow 0\), provided that \(V_0/\mu \rightarrow 16 \, k^2 / (4 \pi \epsilon_0 E)\). In this limit, the Born differential cross-section becomes

\[
\frac{d\sigma}{d\Omega} \simeq \left(\frac{Z \, Z' \, e^2}{16 \pi \epsilon_0 E}\right)^2 \frac{1}{\sin^4(\theta/2)},
\]

where \(E = p^2 / 2m\) is the kinetic energy of the incident particles. Of course, Equation (e17.46) is identical to the famous Rutherford scattering cross-section formula of classical physics.

The Born approximation is valid provided that \(|\psi(\bf r)|\) is not too different from \(|\psi_0(\bf r)|\) in the scattering region. It follows, from Equation (e15.9), that the condition for \(|\psi(\bf r)| \simeq |\psi_0(\bf r)|\) in the vicinity of \(\bf r = 0\) is

\[
\left| \frac{m}{2\pi \hbar^2} \int \frac{\exp(\,i\, k \, r')}{r'} \, V(\bf r') \, d^3r' \right| \ll 1.
\]

Consider the special case of the Yukawa potential. At low energies, \(k \ll \mu\) we can replace \(\exp(\,i\, k \, r')\) by unity, giving

\[
\frac{2m}{\hbar^2} \frac{|V_0|}{\mu^2} \ll 1,
\]

as the condition for the validity of the Born approximation. The condition for the Yukawa potential to develop a bound state is

\[
\frac{2m}{\hbar^2} \frac{|V_0|}{\mu} \geq 2.7,
\]

where \(V_0\) is negative. Thus, if the potential is strong enough to form a bound state then the Born approximation is likely to break down. In the high-\(k\) limit, Equation (e17.47) yields

\[
\frac{2m}{\hbar^2} \frac{|V_0|}{\mu k} \ll 1.
\]

This inequality becomes progressively easier to satisfy as \(k\) increases, implying that the Born approximation is more accurate at high incident particle energies.

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