14.2: Born Approximation

Equation ([e15.17]) is not particularly useful, as it stands, because the quantity \( f({\bf k}, {\bf k}') \) depends on the, as yet, unknown wavefunction \( \psi({\bf r}) \). [See Equation ([e5.12]).] Suppose, however, that the scattering is not particularly strong. In this case, it is reasonable to suppose that the total wavefunction, \( \psi({\bf r}) \), does not differ substantially from the incident wavefunction, \( \psi_0({\bf r}) \). Thus, we can obtain an expression for \( f({\bf k}, {\bf k}') \) by making the substitution \( \psi({\bf r}) \rightarrow \psi_0({\bf r}) = \sqrt{n} \exp(i {\bf k} \cdot {\bf r}) \) in Equation ([e5.12]).

This procedure is called the Born approximation.

The Born approximation yields
\[
f({\bf k}, {\bf k}') \simeq \frac{m}{2\pi \hbar^2} \int e^{i({\bf k} - {\bf k}') \cdot {\bf r}'} V({\bf r}') d^3{\bf r}'.
\]
Thus, \( f({\bf k}, {\bf k}') \) becomes proportional to the Fourier transform of the scattering potential \( V({\bf r}) \) with respect to the wavevector \( q = |{\bf k} - {\bf k}'| \).

For a spherically symmetric potential,
\[
f({\bf k}', {\bf k}) \simeq -\frac{2m}{\hbar^2 q} \int_0^\infty r' V(r') \sin(q r') dr'.
\]

Consider scattering by a Yukawa potential,
\[
V(r) = \frac{V_0 \exp(-\mu r)}{\mu r},
\]
where \( V_0 \) is a constant, and \( 1/\mu \) measures the “range” of the potential. It follows from Equation ([e17.38]) that
\[
f(\theta) = -\frac{2m V_0}{\hbar^2 \mu} \frac{1}{q^2 + \mu^2},
\]
where \( q \) is the scattering angle. Recall that the vectors \( {\bf k} \) and \( {\bf k}' \) have the same length, via energy conservation.
The Yukawa potential reduces to the familiar Coulomb potential as \( \mu \to 0 \), provided that \( V_0 / \mu \to Z \angle Z' \angle \epsilon / (4 \pi \epsilon_0) \). In this limit, the Born differential cross-section becomes \( \frac{d\sigma}{d\Omega} \simeq \left( \frac{2mZZ'\epsilon^2}{16\pi\epsilon_0E} \right)^2 \frac{1}{\sin^4(\theta/2)} \). Recalling that \( \hbar k \) is equivalent to \( |\mathbf{p}| \), the previous equation can be rewritten \( \frac{d\sigma}{d\Omega} \simeq \left( \frac{Z \angle Z' \angle \epsilon^2}{16\pi\epsilon_0E} \right)^2 \frac{1}{\sin^4(\theta/2)} \), where \( E = p^2 / 2m \) is the kinetic energy of the incident particles. Of course, Equation (e17.46) is identical to the famous Rutherford scattering cross-section formula of classical physics.

The Born approximation is valid provided that \( \psi(\mathbf{r}) \) is not too different from \( \psi_0(\mathbf{r}) \) in the scattering region. It follows, from Equation (e15.9), that the condition for \( \psi(\mathbf{r}) \simeq \psi_0(\mathbf{r}) \) in the vicinity of \( \mathbf{r} = 0 \) is \( \left| \frac{m}{2\pi\hbar^2} \int \frac{\exp(\mathbf{i}k\mathbf{r'})}{\mathbf{r'}} V(\mathbf{r'}) \,d^3\mathbf{r'} \right| \ll 1 \). Consider the special case of the Yukawa potential. At low energies, (i.e., \( k \ll \mu \)) we can replace \( \exp(\mathbf{i}k\mathbf{r'}) \) by unity, giving \( \frac{2m}{\hbar^2} \frac{|V_0|}{\mu^2} \ll 1 \) as the condition for the validity of the Born approximation. The condition for the Yukawa potential to develop a bound state is \( \left| \frac{2m}{\hbar^2} \frac{|V_0|}{\mu k} \right| \ll 1 \). This inequality becomes progressively easier to satisfy as \( k \) increases, implying that the Born approximation is more accurate at high incident particle energies.

- Richard Fitzpatrick (Professor of Physics, The University of Texas at Austin)