4.7: Simple Harmonic Oscillator

The classical Hamiltonian of a simple harmonic oscillator is \[ H = \frac{p^2}{2m} + \frac{1}{2}Kx^2, \] where \( K > 0 \) is the so-called force constant of the oscillator. Assuming that the quantum mechanical Hamiltonian has the same form as the classical Hamiltonian, the time-independent Schrödinger equation for a particle of mass \( m \) and energy \( E \) moving in a simple harmonic potential becomes

\[ \frac{d^2\psi}{dx^2} = \frac{2m}{\hbar^2} \left( \frac{1}{2}Kx^2 - E \right) \psi. \] 

Let \( \omega = \sqrt{K/m} \), where \( \omega \) is the oscillator’s classical angular frequency of oscillation. Furthermore, let \( y = \sqrt{\frac{m\omega}{\hbar}}x \), and \( \epsilon = \frac{2E}{\hbar\omega} \). Equation (5.90) reduces to

\[ \frac{d^2\psi}{dy^2} - (y^2 - \epsilon)\psi = 0. \]

We need to find solutions to the previous equation which are bounded at infinity: that is, solutions which satisfy the boundary condition \( \psi \to 0 \) as \( |y| \to \infty \).

Consider the behavior of the solution to Equation (5.93) in the limit \(|y| \gg 1| \). As is easily seen, in this limit the equation simplifies somewhat to give

\[ \frac{d^2\psi}{dy^2} - y^2\psi \approx 0. \]

The approximate solutions to the previous equation are \( \psi(y) \approx A(y)e^{\pm y^2/2} \), where \( A(y) \) is a relatively slowly varying function of \( y \). Clearly, if \( \psi(y) \) is to remain bounded as \( |y| \to \infty \), then we must choose the exponentially decaying solution. This suggests that we should write \( \psi(y) = h(y)e^{-y^2/2} \), where we would expect \( h(y) \) to be an algebraic, rather than an exponential, function of \( y \).

Substituting Equation (5.96) into Equation (5.93), we obtain

\[ \frac{d^2h}{dy^2} - 2y\frac{dh}{dy} + (\epsilon - 1)h = 0. \]

Let us attempt a power-law solution of the form \( h(y) = \sum_{i=0}^{\infty} c_i y^{2i} \), where \( c_{-1} = 0 \). Inserting this test solution into Equation (5.97), and equating the coefficients of \( y^{2i} \), we obtain the recursion relation \( c_{i+2} = \frac{(2i - \epsilon + 1)}{(i+1)(i+2)} c_i \).

Consider the behavior of \( h(y) \) in the limit \(|y| \gg \infty \). The previous recursion relation simplifies to \( c_{i+2} \approx \frac{2}{i} c_i \). Hence, at large \(|y|\), when the higher powers of \( y \) dominate, we have \( h(y) \approx C \sum_{j} \frac{y^{2j}}{j!} \) for some constant \( C \).
C,\{\text{e}\}^{y^2}.\] It follows that \(\psi(y) = h(y)\exp(-y^2/2)\) varies as \(\exp(y^2/2)\) as \(|y|\rightarrow\infty\). This behavior is unacceptable, because it does not satisfy the boundary condition \(\psi(\rightarrow 0) = 0\) as \(\exp(y^2/2)\). The only way in which we can prevent \(\psi\) from blowing up as \(\exp(y^2/2)\) is to demand that the power series ([e5.98]) terminate at some finite value of \(\psi\). This implies, from the recursion relation ([e5.99]), that \(|\epsilon| = 2n+1,\) where \(n\) is a non-negative integer. Note that the number of terms in the power series ([e5.98]) is \(n+1\). Finally, using Equation ([e5.92]), we obtain \[E = (n+1/2)\hbar\omega,\] for \(n=0,1,2,\cdots\).

Hence, we conclude that a particle moving in a harmonic potential has quantized energy levels that are equally spaced. The spacing between successive energy levels is \(\hbar\omega\), where \(\omega\) is the classical oscillation frequency. Furthermore, the lowest energy state \((n=0)\) possesses the finite energy \((1/2)\hbar\omega\). This is sometimes called zero-point energy. It is easily demonstrated that the (normalized) wavefunction of the lowest energy state takes the form \[\psi_{0}(x) = \frac{\exp(-x^2/2)}{\pi^{1/4}\sqrt{d}}\] Let \(\psi_{n}(x)\) be an energy eigenstate of the harmonic oscillator corresponding to the eigenvalue \[E_{n} = (n+1/2)\hbar\omega,\] \(n\) being a non-negative integer. \(\psi_{n}(x)\) is then a solution of the Schrödinger equation \[\left(-\frac{d^2}{dx^2} + x^2\right)\psi_{n}(x) = E_{n}\psi_{n}(x).\] We conclude that \(a_+\) and \(a_-\) are raising and lowering operators, respectively, for the harmonic oscillator: that is, operating on the wavefunction with \(a_+\) causes the quantum number \(n\) to increase by unity, and vice versa. The Hamiltonian for the harmonic oscillator can be written in the form \[H = \frac{\hbar\omega}{2}a_+a_- + \frac{\hbar\omega}{2},\] from which the result \[H\psi_n = (n+1/2)\hbar\omega\psi_n\] is readily deduced. Finally, Equations ([e5.107]), ([e5.113]), and ([e5.114]) yield the useful expression \[\left(\frac{d^2}{dx^2} + x^2\right)\psi_{m}(x) = \frac{\hbar\omega}{2}a_+a_- + \frac{\hbar\omega}{2}\psi_{m}(x).\] Contributors and Attributions

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