12.11: 2P-1S Transitions in Hydrogen

Let us calculate the rate of spontaneous emission between the first excited state (i.e., \(n=2\)) and the ground-state (i.e., \(n'=1\)) of a hydrogen atom. Now, the ground-state is characterized by \(l'=m'=0\). Hence, in order to satisfy the selection rules (\([e13.133]\) and \([e13.134]\)), the excited state must have the quantum numbers \(l=1\) and \(m=0, \pm 1\). Thus, we are dealing with a spontaneous transition from a \(2P\) to a \(1S\) state. Note, incidentally, that a spontaneous transition from a \(2S\) to a \(1S\) state is forbidden by our selection rules.

According to Section \([s10.4]\), the wavefunction of a hydrogen atom takes the form
\[
\psi_{n,l,m}(r,\theta,\phi) = R_{n,l}(r) Y_{l,m}(\theta,\phi),
\]
where the radial functions \(R_{n,l}\) are given in Section \([s10.4]\), and the spherical harmonics \(Y_{l,m}\) are given in Section \([sharm]\). Some straightforward, but tedious, integration reveals that
\[
\begin{aligned}
\langle 1,0,0|x|2,1,\pm 1 \rangle &= \pm \frac{2^7}{3^5} a_0, \\
\langle 1,0,0|y|2,1,\pm 1 \rangle &= i \frac{2^7}{3^5} a_0, \\
\langle 1,0,0|z|2,1,0 \rangle &= \sqrt{2} \frac{2^7}{3^5} a_0,
\end{aligned}
\]
where \(a_0\) is the Bohr radius specified in Equation \([e9.57]\). All of the other possible \(2P\rightarrow 1S\) matrix elements are zero because of the selection rules. It follows from Equation \([e13.128]\) that the modulus squared of the dipole moment for the \(2P\rightarrow 1S\) transition takes the same value
\[
\begin{aligned}
d^2 &= \frac{2^{15}}{3^{10}} (e a_0)^2, \\
\end{aligned}
\]
for \(m=0\), \(1\), or \(-1\). Clearly, the transition rate is independent of the quantum number \(m\). It turns out that this is a general result.

Now, the energy of the eigenstate of the hydrogen atom characterized by the quantum numbers \((n\,\rangle, \langle l\,\rangle, \langle m\,)\) is \(E = E_0/n^2\), where the ground-state energy \(E_0\) is specified in Equation \([e9.56]\). Hence, the energy of the photon emitted during a \(2P\rightarrow 1S\) transition is \(\hbar \omega = E_0/4 - E_0 = -3/4 E_0 = 10.2\text{ eV}\). This corresponds to a wavelength of \(\lambda = (1.215 \times 10^{-7}) \text{ m} \).

Finally, according to Equation \([e3.115]\), the \(2P\rightarrow 1S\) transition rate is written \(w_{2P\rightarrow 1S} = \)
\[ \frac{\omega^3 d^2}{3 \pi \epsilon_0 \hbar c^3}, \] which reduces to \[ w_{2P \rightarrow 1S} = \left( \frac{2}{3} \right)^8 \alpha^5 \frac{m_e c^2}{\hbar} = 6.27 \times 10^8 \text{ s}^{-1} \] with the aid of Equations (e13.139) and (e13.140). Here, \( \alpha = 1/137 \) is the fine-structure constant. Hence, the mean life-time of a hydrogen \( 2P \) state is \[ \tau_{2P} = (w_{2P \rightarrow 1S})^{-1} = 1.6 \text{ ns}. \] Incidentally, because the \( 2P \) state only has a finite life-time, it follows from the energy-time uncertainty relation that the energy of this state is uncertain by an amount \[ \Delta E_{2P} \sim \frac{\hbar}{\tau_{2P}} \sim 4 \times 10^{-7} \text{ eV}. \] This uncertainty gives rise to a finite width of the spectral line associated with the \( 2P \rightarrow 1S \) transition. This natural line-width is of order \[ \frac{\hbar \omega}{\tau_{2P}} \sim 4 \times 10^{-8}. \]