12.11: 2P-1S Transitions in Hydrogen

Let us calculate the rate of spontaneous emission between the first excited state (i.e., \(n=2\)) and the ground-state (i.e., \(n'=1\)) of a hydrogen atom. Now, the ground-state is characterized by \(l'=m'=0\). Hence, in order to satisfy the selection rules ([e13.133]) and ([e13.134]), the excited state must have the quantum numbers \(l=1\) and \(m=0,\pm 1\). Thus, we are dealing with a spontaneous transition from a \(2P\) to a \(1S\) state. Note, incidentally, that a spontaneous transition from a \(2S\) to a \(1S\) state is forbidden by our selection rules.

According to Section [s10.4], the wavefunction of a hydrogen atom takes the form \(\psi_{n,l,m}(r,\theta,\phi) = R_{n,l}(r)Y_{l,m}(\theta,\phi)\), where the radial functions \(R_{n,l}\) are given in Section [s10.4], and the spherical harmonics \(Y_{l,m}\) are given in Section [sharm]. Some straightforward, but tedious, integration reveals that

\[
\begin{aligned}
\langle 1,0,0|x|2,1,\pm 1\rangle &= \pm \frac{2^7}{3^5}a_0, \\
\langle 1,0,0|y|2,1,\pm 1\rangle &= i\frac{2^7}{3^5}a_0, \\
\langle 1,0,0|z|2,1,0\rangle &= \sqrt{2}\frac{2^7}{3^5}a_0,
\end{aligned}
\]

where \(a_0\) is the Bohr radius specified in Equation ([e9.57]). All of the other possible \(2P\rightarrow 1S\) matrix elements are zero because of the selection rules. It follows from Equation ([e13.128]) that the modulus squared of the dipole moment for the \(2P\rightarrow 1S\) transition takes the same value \(\langle\text{label}\,\text{e13.139}\rangle\)

\[
d^2 = \frac{2^{15}}{3^{10}}(e\,a_0)^2
\]

for \(m=0\), \(1\), or \((-1)\). Clearly, the transition rate is independent of the quantum number \(m\). It turns out that this is a general result.

Now, the energy of the eigenstate of the hydrogen atom characterized by the quantum numbers \(\langle n\rangle, \langle l\rangle, \langle m\rangle\) is \(E = E_0/n^2\), where the ground-state energy \(E_0\) is specified in Equation ([e9.56]). Hence, the energy of the photon emitted during a \(2P\rightarrow 1S\) transition is \(\langle\text{label}\,\text{e13.140}\rangle\)

\[
\hbar\omega = E_0/4 - E_0 = -\frac{3}{4}E_0 = 10.2\text{ eV}.
\]

This corresponds to a wavelength of \((1.215\times 10^{-7})\) m.

Finally, according to Equation ([e3.115]), the \(2P\rightarrow 1S\) transition rate is written \(\langle w_{2P\rightarrow 1S} = \)
\[
\frac{\omega^3 d^2}{3 \pi \epsilon_0 \hbar c^3},
\]
which reduces to
\[
w_{2P \rightarrow 1S} = 6.27 \times 10^8 \text{ s}^{-1}
\]
with the aid of Equations (e13.139) and (e13.140). Here, \(\alpha = 1/137\) is the fine-structure constant. Hence, the mean life-time of a hydrogen \(\text{(2P)}\) state is
\[
\tau_{2P} = \frac{1}{w_{2P \rightarrow 1S}} = 1.6 \text{ ns}.
\]
Incidentally, because the \(\text{(2P)}\) state only has a finite life-time, it follows from the energy-time uncertainty relation that the energy of this state is uncertain by an amount
\[
\Delta E_{2P} \approx \frac{\hbar}{\tau_{2P}} \approx 4 \times 10^{-7} \text{ eV}.
\]
This uncertainty gives rise to a finite width of the spectral line associated with the \(\text{(2P \rightarrow 1S)}\) transition. This natural line-width is of order
\[
\frac{\Delta \lambda}{\lambda} \sim \frac{\Delta E_{2P}}{\hbar \omega} \sim 4 \times 10^{-8}
\]