14.7: Low-Energy Scattering

In general, at low energies (i.e., when \(1/k\) is much larger than the range of the potential), partial waves with \(l>0\) make a negligible contribution to the scattering cross-section. It follows that, at these energies, with a finite range potential, only \(S\)-wave scattering is important.

As a specific example, let us consider scattering by a finite potential well, characterized by \(V=V_0\) for \(r<a\), and \(V=0\) for \(r\geq a\). Here, \(V_0\) is a constant. The potential is repulsive for \(V_0>0\), and attractive for \(V_0<0\). The outside wavefunction is given by [see Equation \((e17.80)\)]

\[
\begin{aligned}
\mathcal{R}_0(r) &= \exp\left(\text{i} \delta_0\right) \left[ \cos \delta_0 j_0(kr) - \sin \delta_0 y_0(kr) \right] \\
&= \frac{\exp(\text{i} \delta_0) \sin(kr + \delta_0)}{kr},
\end{aligned}
\]

where use has been made of Equations \((e17.58a)\) and \((e17.58b)\). The inside wavefunction follows from Equation \((e17.85)\). We obtain \(\mathcal{R}_0(r) = B \frac{\sin(k'r)}{r}\), where use has been made of the boundary condition \((e17.86)\). Here, \(B\) is a constant, and \(E - V_0 = \hbar^2 k'^2 / 2m\).

For \(E>V_0\), we have \(\mathcal{R}_0(r) = B \frac{\sinh(\kappa r)}{r}\), where \(\kappa = \sqrt{ \frac{2m |V_0|}{\hbar^2} }\). Matching \(\mathcal{R}_0(r)\), and its radial derivative, at \(r=a\) yields \(\tan(k'a + \delta_0) = \frac{k}{k'} \tan(k'a)\) for \(E>V_0\), and \(\tan(k'a + \delta_0) = \frac{k}{\kappa} \tanh(\kappa a)\) for \(E<V_0\).

Consider an attractive potential, for which \(E>V_0\). Suppose that \(|V_0|\gg E\) (i.e., the depth of the potential well is much larger than the energy of the incident particles), so that \(k'\gg k\). We can see from Equation \((e17.107)\) that, unless \(\tan(k'a)\) becomes extremely large, the right-hand side is much less that unity, so replacing the tangent of a small quantity with the quantity itself, we obtain \(\kappa + \delta_0 \approx \frac{k}{k'} \tan(k'a)\). This yields \(\delta_0 \approx k'a \left[ \frac{\tan(k'a)}{k'a} - 1 \right]\). According to Equation \((e17.99)\), the scattering cross-section is given by

\[
\sigma_{\text{total}} \approx \frac{4\pi}{k^2} \sin^2 \delta_0 = 4\pi a^2 \left[ \frac{\tan(k'a)}{k'a} - 1 \right]^2.
\]
sufficiently small values of \( (k', a) \), \( k' \, a \approx \sqrt{\frac{2 \, m \, |V_0| \, a^2}{\hbar^2}} \). It follows that the total (\( S \)-wave) scattering cross-section is independent of the energy of the incident particles (provided that this energy is sufficiently small).

Note that there are values of \( (k', a) \) (e.g., \( (k', a) \approx 4.49 \)) at which \( \delta_0 \to \pi \), and the scattering cross-section \( \langle e17.111 \rangle \) vanishes, despite the very strong attraction of the potential. In reality, the cross-section is not exactly zero, because of contributions from \( (l>0) \) partial waves. But, at low incident energies, these contributions are small. It follows that there are certain values of \( (V_0) \) and \( (k') \) that give rise to almost perfect transmission of the incident wave. This is called the Ramsauer-Townsend effect, and has been observed experimentally.

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**Contributors and Attributions**

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