1.1: Complex Numbers

**Definitions and Properties**

A *complex number* consists of a combination of a *real part* and an *imaginary part*, the former being a real number and the latter multiplying \(\sqrt{-1}\), which we denote as "\(i\)."

\[z=a+bi, \quad a \equiv \text{Re}(z), \quad b \equiv \text{Im}(z)\]

A strictly real or imaginary number is also complex, with the imaginary or real part equal to zero, respectively. Complex numbers form what is called a *field* in mathematics, which (in a nutshell – this is not a text in pure mathematics) means that:

- products and sums of complex numbers are also complex numbers
- products and sums satisfy the usual associative/commutative/distributive properties of real numbers
- all of them have inverses except for zero

These are all properties of real numbers, but note that operations like square roots of complex numbers also produce complex numbers, while this is not true of real numbers.

The *complex conjugate* of a complex number is another complex number where the sign of the imaginary part of the original number is flipped. The operation of *complex conjugation* is easily achieved by changing all of the \(i\)'s in the expression of the complex number to \((-i)\). The complex conjugate of a variable representing a complex number is denoted with a star superscript:

\[z=a+bi; \quad z^* =a-bi\]
The magnitude or modulus of a complex number is the (positive) square-root of the product of the number and its complex conjugate:

\[
|z|^2 = z \cdot z^* = (a + bi)(a - bi) = a^2 + b^2
\]

**Argand Diagrams**

As Equation 1.1.3 suggests, we can express a complex number as a vector in a plane, though to distinguish these from vectors, they are typically given the name *phasor*, for reasons that will become clear shortly. The magnitude of such an object would then be the length of the phasor, with the components being the real and imaginary parts. Each point in this real/imaginary plane (as well as the phasor that points to it from the origin) corresponds to a unique complex number. This graphical representation is known as an *Argand diagram*.

![Argand Diagram](Figure 1.1.1 – Argand Diagram)

**The Harmonic Connection**

The Argand diagram representation of complex numbers becomes particularly enlightening when we look at it in polar coordinates. It is customary to define \(\theta\) as the angle between the phasor and the positive \(x\)-axis. If we call the magnitude of the complex number \(R = \sqrt{a^2 + b^2}\), then the real and imaginary "components" are:

\[
\begin{align*}
a &= R \cos \theta, \\
b &= R \sin \theta, \\
\end{align*}
\]

The equation of motion of an object following simple harmonic motion can be written as either the real or imaginary part of a phasor that is rotating at a constant rate. The amplitude of motion is the magnitude of the phasor, and the phase of the sine function that describes its motion is the angle the phasor makes with the real axis, \(\theta\). Physical harmonic waves exhibit harmonic motion at each fixed point in space, so they too can be described in terms of components of phasors. As we will see, in quantum mechanics, waves are expressed in terms of the full phasor (rather than just a single component).

Rather than always writing complex numbers in the form \(z = R \cos \theta + i R \sin \theta\), it is convenient to employ...
the famous Euler identity:

\[
\begin{align*}
&e^{i\theta} = \cos \theta + i \sin \theta \\
\Rightarrow & \quad z = Re^{i\theta}
\end{align*}
\]

Note that this format allows for easy proof of Equation 1.1.3:

\[
|z|^2 = \sqrt{z \cdot z^*} = \sqrt{[Re^{i\theta}] [Re^{-i\theta}]} = \sqrt{R^2 e^{i\theta-i\theta}} = R
\]

It is also useful to know the inverse relationships, which can be derived with some quick algebra:

\[
\begin{align*}
\cos \theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} \\
\sin \theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i}
\end{align*}
\]

Example \(\PageIndex{1}\)

Not surprisingly, phasors add exactly like vectors. Use this fact to prove the following trig identity:

\[
1 + \cos \left[\left(1\right) \left(\frac{2\pi}{n}\right)\right] + \cos \left[\left(2\right) \left(\frac{2\pi}{n}\right)\right] + \ldots + \cos \left[\left(n-1\right) \left(\frac{2\pi}{n}\right)\right] = 0 \nonumber
\]

Solution

If we construct \(n\) phasors of magnitude 1 with angles equally-spaced around the full \(2\pi\) range, then adding them like vectors will result in a zero "net phasor." Placing the first phasor at \(\theta = 0\), it is just equal to 1. The second phasor's angle is \(\frac{2\pi}{n}\). We keep adding phasors at angles bigger than the previous one by \(\frac{2\pi}{n}\), until we get all the way around (and we don't repeat the first phasor at \(\theta = 2\pi\)). The sum of these is zero, so:

\[
1 + e^{\frac{2\pi i}{n}} + e^{\frac{4\pi i}{n}} + \ldots + e^{\frac{(n-1)2\pi i}{n}} = 0 \nonumber
\]

This is a sum of complex numbers, which means that the sums of the real parts are zero. The real parts are the cosines of these angles, proving the identity. Note that the imaginary part is also zero, proving the same identity for the sines of those angles.