3. Harmonic Waves

Harmonic Wave Equation

For the rest of the course we will focus on infinite repeating waves of a specific type: harmonic waves. Mechanical harmonic waves can be expressed mathematically as:

\[ y(x,t) - y_0 = A \sin\left( 2 \pi \frac{t}{T} \pm 2 \pi \frac{x}{\lambda} + \phi \right) \]

The displacement of a piece of the wave at equilibrium position \((x)\) and time \((t)\) is given by the whole left hand side \((y(x,t) - y_0)\). \((y_0)\) is the position of the medium without any wave, and \((y(x,t))\) is its actual position. Earlier we carelessly used \((y(x,t))\) to describe the displacement, but to be precise we must describe the displacement using the entire left side.

On the right hand side, we're familiar with \(A\) the amplitude, \(T\) the period, and \(\lambda\) the wavelength. The fixed phase constant \(\phi\) is new; it describes what the wave looks like at \((x=0,t=0))\). The symbol \(\pm\) asks us to choose \(+\) or \(-\), which describes the direction that the wave travels.

Before going too much further, it is worthwhile noting the difference between variables and parameters.

- The parameters \((A, T, \lambda, \phi, y_0)\) and the choice of \((+\) or \((-\) are defined for any given harmonic wave. They describe the wave and its behavior.
- The wave exists in all space and at any time (for any \((x)\) and any \((t)\)). In the formula above, \((x)\) and \((t)\) are chosen by us to answer a question about the displacement of a specific piece in the medium at a specific time. This distinction makes \((x)\) and \((t)\) variables.

In other words, we can ask about different locations and times by changing variables, but parameters for a wave are fixed values.
Extending the Harmonic Wave Model

While we have framed this discussion in terms of material waves because it is the easiest to visualize, we should be aware that the harmonic wave is a much more general concept. It can apply to the variation in pressure (for sound waves):

\[ P(x,t) - P_0 = A \sin \left( 2\pi \frac{t}{T} \pm 2\pi \frac{x}{\lambda} + \phi \right) \]

or the variation in electric field due to a light wave.

\[ \textbf{E}(x,t) - \textbf{E}_0 = A \sin \left( 2\pi \frac{t}{T} \pm 2\pi \frac{x}{\lambda} + \phi \right) \]

In general we can let \( y(x,t) \) stand for any of these physical quantities, not just position. We shall refer to \( y(x,t) \) in this general form as the wave function. Sometimes harmonic waves are also called sinusoidal waves as the wave function represents a sine or cosine function.

While waves in the real world do not go on forever, and do not exist for all time, we can still use harmonic waves of this form as a good approximation. They offer a considerable simplification.

Total Phase \( \Phi(x,t) \)

If we wanted to, we could define a new quantity \( \Phi(x,t) \) as a function of \( x \) and \( t \) so that

\[ \Phi(x,t) = 2\pi \frac{t}{T} \pm 2\pi \frac{x}{\lambda} + \phi \]

Using this, we can rewrite our general harmonic wave formula as

\[ y(x,t) - y_0 = A \sin \Phi(x,t) \]

Because the sine function is periodic with period \( 2\pi \), changing \( \Phi(x,t) \) by \( 2\pi, 4\pi, \ldots \) does not change \( y(x,t) \). This ambiguity exists partially because the wave repeats so that many places on the wave look exactly the same. Let us try and make our example more concrete: because \( \sin \frac{\pi}{2} = 1 \) is the maximum of the sine function, \( \Phi = \frac{\pi}{2} \) labels a peak in our wave. Note that \( \Phi = \frac{5\pi}{2} \) and \( \Phi = \frac{3\pi}{2} \) also label peaks, but they label different peaks. When we imagine ourselves riding the wave, or when we watch a wave peak travel, we are really following a point of constant total phase. The next example should make this more clear.

Example \( \PageIndex{1} \)

We are going to look at the wave described by this equation
\[ y(x,t) = (25 \text{ cm}) \sin \left( 2 \pi \frac{t}{4 \text{ s}} + 2 \pi \frac{x}{4 \text{ cm}} + \frac{\pi}{2} \right) \]

a. One of the peaks of the wave has a total phase \( \Phi = \frac{\pi}{2} \). What is the location of this peak when \( t = 0, t = 1 \text{ s}, \ t = 2 \text{s}, \ t = 3 \text{s}, \) and \( t = 4 \text{ s} \)?

b. Is the wave travelling to the left or right?

**Solution**

**a.** We only need information about the total phase to solve this. We are given

\[ \Phi(x,t) = 2 \pi \frac{t}{4 \text{ s}} + 2 \pi \frac{x}{4 \text{ cm}} + \frac{\pi}{2} \]

We are asked to find where (for which \( x \)) \( \Phi = \frac{\pi}{2} \) is at \( t = 0 \). We can solve this rather simply:

\[ \Phi(x,t) = \frac{\pi}{2} = 2 \pi \frac{0}{4 \text{ s}} + 2 \pi \frac{x}{4 \text{ cm}} + \frac{\pi}{2} \]

Which is only satisfied for \( x = 0 \). Therefore the \( \Phi = \frac{\pi}{2} \) peak is at \( x = 0 \) when \( t = 0 \). Substituting the remaining values for \( t \), we find that for \( t = 1 \text{ s}, 2 \text{ s}, 3 \text{ s}, \) and \( 4 \text{ s} \), we have the peak is located at \( x = 0 \text{ cm}, -1 \text{ cm}, -2 \text{ cm}, -3 \text{ cm}, \) and \( -4 \text{ cm} \) respectively.

**b.** We see that a particular peak goes from 0 cm to -4 cm. The peak moves in the direction of negative \( x \), which is to the left by our convention.

We can see that the \( + \) sign in front of the spatial term is responsible for this. Remember that by riding the wave we are actually looking at a piece of constant total phase \( \Phi(x, t) \). Going back to our equation, to ensure the left side of our equation remains constant as \( \Phi(x, t) \) increases, another term must decrease. \( \phi \) the phase constant of our wave does not change with time. Therefore our \( x \) term must decrease, showing again that our wave travels to the left.

**Exercise**

Carry out Example #2 on your own using the \( - \) sign instead.

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**Fixed Phase Constant \( \phi \)**

Note that the phase expression is very similar to the mathematical description we developed for the motion of a particle vibrating in simple harmonic motion. The first term in the argument of the sine function, \( 2 \pi \frac{t}{T} \), like in harmonic motion, gives information about the phase for different values of \( t \). The fixed phase constant \( \phi \) gives the proper value of \( y \) at \( t = 0 \) and \( x = 0 \). The new term, \( 2 \pi \frac{x}{\lambda} \), resembles the time-dependent term very closely. This term involving \( x \) and \( \lambda \) gives the change in phase as we look along different values of \( x \).

The total phase \( \Phi(x) \) goes through a complete cycle of \( 2 \pi \) radians each time \( x \) increases or decreases by an amount equal to the wavelength \( \lambda \). Likewise the total phase \( \Phi \) goes through a complete cycle of \( 2 \pi \) radians each time \( t \) increases by one period \( T \). This is a reminder that \( \lambda \) controls repetition in space, while \( T \) controls repetition in...
Note that if \( y=0 \) and \( t=0 \), we have \( \Phi'=\phi' \); the total phase is given by the fixed phase constant.

**Relationship between \( v_{\text{wave}} \), \( \lambda \), and \( f \)**

We have already learned that the speed of the wave depends on the properties of the medium. When dealing with repeating waves we must consider three additional parameters: the wavelength \( \lambda \), the period \( T \) and the frequency \( f \). Recall that frequency and period are related by \( f=1/T \), so only two of the three parameters are independent. Below, we go through two different arguments to show that these parameters are related to wave speed \( v_{\text{wave}} \) by

\[
v_{\text{wave}} = \lambda f.
\]

### One Argument, Distance over Time

Our definition of speed (more specifically, velocity) from Physics 7B can be written as

\[
\text{speed} = \frac{\text{Distance travelled}}{\text{Time spent}}
\]

Let us look at the wave at a particular time, and focus on a particular peak indicated by the solid dot.

Recall that one period is the shortest amount of time before the wave looks *exactly* the same. For the wave to look exactly the same, the peak indicated by the solid black dot must have moved to the location of the dashed circle, which means it moved a distance of one wavelength. We can now calculate the speed of the peak (which is the same as the speed of the entire wave).

\[
v_{\text{wave}} = \frac{\text{Distance}}{\text{Time}} = \frac{\lambda}{T}
\]

Using the fact that \( 1/T \) is another way of writing the frequency \( f \), we can write this formula in a more familiar form

\[
v_{\text{wave}} = \lambda f.
\]

Note that this method works for a wave traveling in the opposite direction as well. We would still see a peak travel one wavelength in one period, so we would still obtain \( v_{\text{wave}} = \lambda f \).
Another Argument, Following the Total Phase

A superficially different way of finding the wave speed is to follow a piece of the wave with a constant total phase $\Phi$ (see Example). Let us pick a phase and look at it at two different times $t_1$ (where it is at $x_1$) and $t_2$ (where that piece of the wave is at $x_2$). This gives us the relationships

$$\Phi = 2\pi \frac{t_1}{T} \pm 2\pi \frac{x_1}{\lambda} + \phi$$
$$\Phi = 2\pi \frac{t_2}{T} \pm 2\pi \frac{x_2}{\lambda} + \phi$$

Now we can subtract these equations from one another and rewrite them as follows:

$$2\pi \left(\frac{t_2 - t_1}{T}\right) \pm 2\pi \left(\frac{x_1 - x_2}{\lambda}\right) = 0$$

where we use $\Delta$ to mean "final minus initial." Dividing the whole equation by $2\pi$ and rearranging, we get

$$\frac{\Delta x}{\Delta t} = \mp \frac{\lambda}{T}$$

But this expression tells us how far ($\Delta x$) the “disturbance” with phase $\Phi$ moved in a time $\Delta t$. This is exactly what we mean by velocity! Taking the absolute value of this gives us the wave speed:

$$v_{wave} = \left| \frac{\text{distance traveled}}{\text{time taken}} \right| = \left| \frac{\Delta x}{\Delta t} \right| = \frac{\lambda}{T} = \lambda f$$

Contributors

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