2. Superposition of Harmonic Waves

While the idea of superposition is relatively straightforward, actually adding the displacements of the waves at every point for all time is a lot of tedious work. We are now going to specialize superposition to the interference of two infinite harmonic waves with the same frequency. Instead of keeping track of both the wave functions \( \Delta y_1 \) and \( \Delta y_2 \) this means that we only have to look at the difference in total phase \( \Delta \Phi \).

For example if we know that at a particular location the peaks of both waves arrive simultaneously, and the troughs of both waves are occurring simultaneously then we would say the waves are in phase. Our obvious guess would be that \( \Delta \Phi \equiv \Phi_2 - \Phi_1 = 0 \) because the peaks and the troughs are arriving together. However we know that if the total phase changes by \( 2 \pi, 4 \pi, 6 \pi, \ldots \) then the wave will look exactly the same (this is because the sine function repeats over intervals of \( 2 \pi \)). If we have constructive interference all we know is that \( \Delta \Phi \) could be \( 2 \pi \) or \( -2 \pi \) or \( 4 \pi \)...

To see what \( \Delta \Phi \) tells us about the type of interference that occurs, it helps to recall two important characteristics of the sine function:

- As stated above, the sine function is periodic, so \( \sin(\Phi) = \sin(\Phi + 2 \pi n) \) where \( n \) is any integer.

- \( \sin(\Phi + \pi) = -\sin(\Phi) \). In other words, changing \( \Phi \) by an amount \( \pi \) has the same effect as multiplying the equation by -1. The same result holds if we replace \( \pi \) by \( 3 \pi, 5 \pi, -\pi, \ldots \)

Recalling that the total displacement is given by \( \Delta y_{\text{total}}(x,t) = A_1 \sin(\Phi_1) + A_2 \sin(\Phi_2) \) we see that when the sines are the same (when \( \Phi_1 - \Phi_2 = \Delta \Phi = 0, 2 \pi, 4 \pi, \ldots \)) we have constructive interference. when the sines have opposite signs (when \( \Phi_1 - \Phi_2 = \pi, 3 \pi, 5 \pi, \ldots \)) we have destructive interference. Anything else is partial interference.
<table>
<thead>
<tr>
<th>Interference Type</th>
<th>$\Delta \Phi =$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constructive</td>
<td>Even multiples of pi: (0, 2\pi, 4\pi, -2\pi \ldots)</td>
</tr>
<tr>
<td>Destructive</td>
<td>Odd multiples of pi: (\pi, 3\pi, 5\pi, -\pi \ldots)</td>
</tr>
<tr>
<td>Partial</td>
<td>Other</td>
</tr>
</tbody>
</table>

Recall that the total phase \(\Phi(x, t)\) for each wave depends on both \(x\) and \(t\), so \(\Delta \Phi\) can also depend on both \(x\) and \(t\). Strictly speaking we should not talk about whether two waves have constructive, destructive or partial interference, but rather if two waves at a specific location, at a specific time, have constructive, destructive or partial interference.

To keep track of all the terms that can contribute to the change in phase, we introduce the phase chart. The phase chart does not contain more information than the three equations

$$\Phi_1 = 2\pi \frac{t}{T_1} \pm 2\pi \frac{x_1}{\lambda_1} + \phi_1$$

$$\Phi_2 = 2\pi \frac{t}{T_2} \pm 2\pi \frac{x_2}{\lambda_2} + \phi_2$$

$$\Delta \Phi = \Phi_2 - \Phi_1$$

but it is meant to remind you to think about each term. The phase chart is shown below:

<table>
<thead>
<tr>
<th></th>
<th>(2\pi \frac{t}{T})</th>
<th>(\pm 2\pi \frac{x}{\lambda})</th>
<th>(\phi)</th>
<th>(\Phi)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wave 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Wave 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Change</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

It is the lower-right hand corner of this chart in the box, marked \(\Delta \Phi\), that determines if the interference is constructive, destructive or partial.

To build our intuition we are going to look at simplified examples and study the effect of each effect individually. In the next three subsections we will study:

- **The effect of changing path length**: By keeping the sources creating waves in phase and at the same frequency, we can study what the effect is of moving one source around.

- **The effect of changing phase or synchronization**: By keeping the same frequency and amplitude, but now allowing the sources to create waves that are not in phase, we can study how out of sync sources affect
interference.

- **Beats**: Looking at the interference of two waves with different frequencies.

There are a couple of other general comments to make. The first is that because we are combining waves in the same place, the waves must be in the same medium. Therefore the two waves have the same wave speed \(v_{\text{wave}}\). Because they have the same wave speed and same frequency, they must also have the same period \(T = 1/f\) and the same wavelength \(\lambda = v_{\text{wave}}/f\). The quantities which may be different are the distance from the source to the detector \(x\), and the phase constant \(\phi\). By changing either of these quantities we can accomplish either constructive or destructive interference.

### Path Length Difference

For this part of the notes we will assume that the two waves are at the same frequency and have the same amplitude. We are also going to assume that the two sources are in phase with one another. The most important assumption is that the frequencies are the same, and we should discuss the consequences of this assumption before doing anything else.

By having the same frequency we know that the waves both have the same period \((T = 1/f)\). Thus, if the waves started oscillating "in phase" or "out of phase," they will stay in phase or out of phase for all \(t\). In other words, if both harmonic waves have the same frequency, then the type of interference depends on where you are, but unlike the completely general case does not depend on when you ask about the type of interference.

We can obtain this result from the phase chart by inserting the information we just discussed in to the time component column for each wave. We see that the time components of each wave are the same, so the they do not contribute anything to the change in total phase:

<table>
<thead>
<tr>
<th></th>
<th>(2\ \pi \left(\frac{t}{T}\right))</th>
<th>(2\ \pi \left(\frac{x}{\lambda}\right))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wave 1</td>
<td>(2\ \pi \left(\frac{t}{T}\right))</td>
<td></td>
</tr>
<tr>
<td>Wave 2</td>
<td>(2\ \pi \left(\frac{t}{T}\right))</td>
<td></td>
</tr>
<tr>
<td>Change</td>
<td></td>
<td>0</td>
</tr>
</tbody>
</table>

As both our waves are traveling in the same medium we know that \(v_{\text{wave}} = f\ \lambda\) is the same. Because the frequency is the same and the speed \(v_{\text{wave}}\) is the same, both sources must then have the same wavelength \(\lambda\). As we can see, looking at two waves with the same frequency leads to large simplifications.

Let us start with two sources that are creating waves in phase with one another, and located the same distance from the detector. A picture of the situation may look like the one below:
By adding these waves together we find that, at the detector, the total wave has twice the amplitude of either wave alone – *this is constructive interference*.

The waves at the detector look identical if we shift one of the sources one wavelength closer to the detector; this is because after moving a distance of one wavelength, the wave looks exactly the same.

Shifting the source by any integer multiple of wavelengths still leads to constructive interference, as the waves still look identical after shifting:

How does shifting the source by one wavelength affect the change in phase? Assuming waves 1 and 2 are propagating outward (so we may use the − sign) we have $\Delta x = x_1 - x_2 = n\lambda$ where $n$ is an integer; the waves are shifted by $n$ wavelengths. We can then write

\[
\Delta \Phi = -2\pi \dfrac{\Delta x}{\lambda} + \Delta \phi
\]

\[
\Delta \Phi = -2\pi \dfrac{n\lambda}{\lambda} + 0
\]

\[
\Delta \Phi = -(2\pi)n
\]

In the third line we used the fact that the sources were in phase (meaning that they were creating peaks together, and creating troughs together) so $\phi = 0$. The quantity $\Delta x \equiv x_1 - x_2$ tells us how much further wave 1 had to travel to reach the detector than wave 2, and is referred to as the *path length difference*.

By shifting one of the sources half a wavelength closer to the detector, we ensure that every peak in wave 1 coincides with a trough in wave 2. This leads to destructive interference as in the picture below:
Changing the separation by a wavelength (so the total separation is one and a half wavelengths) does not change what the waves look like at the detector, so the waves still interfere destructively.

In fact, it is not difficult to see that having a path length difference of \( \lambda (n + \frac{1}{2}) \), where \( n \) is an integer, will lead to destructive interference. To see this is consistent with our understanding of phase difference we calculate \( \Delta \phi \):

\[
\Delta \Phi = -2 \pi \dfrac{\Delta x}{\lambda} + \Delta \phi
\]

\[
\Delta \Phi = -2 \pi \left( n + \dfrac{1}{2} \right) = -2 \pi n + \pi = \text{(odd)} \pi
\]

Note that \( \Delta \phi = 0 \) still; the waves are still creating peaks (or troughs) at the same time as one another. By changing separation distances, we can create the waves in phase that still exhibit destructive interference when the two waves come together.

It is important to distinguish the separation of the sources and the path length difference. In all of the above examples, these are the same. However, consider two sources separated by half a wavelength, but place the detector equal distances from both sources:
Now even though the sources are separated by $\frac{\lambda}{2}$, the wave from each source must travel exactly the same distance to get to the detector. Therefore the path length difference $\Delta x$ is zero – peaks created at the same time will arrive at the same time and will still show constructive interference. Even though we can think of $x$ as a vector quantity as we did in Physics 7B we don’t need to; the only thing of interest is how far the waves travel from their source.

### Constant Phase Difference

Another way of changing the total phase is ensuring that the two sources are not creating peaks together. This is done by manipulating $\phi_1$ and $\phi_2$, the phase constants. To keep things simple we are again going to assume that the frequencies (and the wavelengths) of the waves produced by the two sources are the same. If we start with two sources in the same location, but make one source create a trough while the other source creates a peak we have destructive interference at the location of the detector:

These two sources are now out of phase. Here the constant phase difference $\Delta \phi = \pi$. We could also manipulate the phase constants to achieve constructive interference by insuring the waves are in phase where $\Delta \phi = 0$ or $2 \pi$.

### Using Phase Charts

Let us see how we can reproduce some of the results we had earlier. Let us look at the case where the two sources had the same phase constant ($\phi_1 = \phi_2$), but the sources were separated by one wavelength:

For the two waves we have \[
\Phi_1 = 2 \pi \left( \frac{t}{T} - \frac{x_1}{\lambda} \right) + \phi_1 \\
\Phi_2 = 2 \pi \left( \frac{t}{T} - \frac{x_1}{\lambda} \right) + \phi_2
\] These waves have the same period, so they must have the same frequency ($T = \frac{1}{f}$). They travel in the same medium, so they must have the same speed $v_{\text{wave}}$ and wavelength $\lambda = v_{\text{wave}}/f$. The total phase difference is then

$$\Delta \Phi = \Phi_1 - \Phi_2 = -\frac{2 \pi}{\lambda} \Delta x$$

By using $\Delta x = \lambda$, from our picture, we get $\Delta \Phi = -2 \pi$, which means the interference is constructive. (If you're given a picture, the easier way of doing this is to look at the waves at the detector. They're clearly in phase, so the
interference must be constructive.

As a second example, let us consider the case where the two waves were out of phase and separated by half a wavelength as shown:

There are now two contributions to our change in total phase. We have $\phi_1 - \phi_2 = \pi$ and $x_1 - x_2 = \lambda / 2$.

The change in phase is now

$$
\Delta \Phi = 2 \pi t \left( \frac{1}{T} - \frac{1}{T} \right) - \frac{2 \pi}{\lambda} (x_1 - x_2) + (\phi - \phi)
$$

$$
= 2 \pi t (0) - \frac{2 \pi}{\lambda} \frac{\lambda}{2} + \pi
$$

$$
= 0 - \pi + \pi = 0
$$

Which gives us constructive interference. A phase chart can be useful in making sure to include every term in our phase difference calculation.

**Limits of Phase Charts**

By using phase charts, we assume that in between the source and the detector, the waves look exactly the same. But there are many real world examples where that is not the case. Consider these waves, their sources separated by 2.5 wavelengths.

We see that we get destructive interference just as we expect. Keeping the same separation, but inserting another medium (shown in blue) can lead to constructive interference at the detector.

This is because the wave travels at a different speed in the new medium, so its wavelength changes value. When it leaves the medium, the waves are back in step. Notice that at the detector the wavelengths are identical: $\lambda_1 = \lambda_2$. In using phase charts, we assume that the two waves have the same wavelength along the whole path; our picture above is a counterexample to this. We can get rid of our assumption if we carefully keep track of the total phase of each wave from one medium to another, or if we use the wave equations directly and dispose of the phase chart.
Is our counterexample relevant to any real world examples? Yes! The subject of thin-film interference is based around light interfering, where one ray goes through two mediums and the other ray only goes through one. Thin film interference is responsible for the pretty colors we see on soap bubbles and in puddles on the street where small amounts of oil sit on the surface. Thin film interference is also responsible for the different colors that are seen reflecting in the surface of pearls (the layers are calcium carbonate and water). Thin film interference finds important applications in photography as well.

Contributors

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