3. What are Matter Waves?

Our definition for a wave has been too stringent – in *What Is A Wave?* we referred to (material) waves as oscillations of a medium about its equilibrium position in time and space. We introduced light as oscillations in the electromagnetic fields, but later discussed the validity of a particle model of light. Here, we still choose to refer to light as a wave because it obeys the principle of superposition. Superposition gives rise to constructive and destructive interference; an example of this is two slit interference which we've discussed earlier.

Light of a given frequency, going through two narrow slits, superposes to give rise to the bright and dark bands as shown in the figure below.

Particles, on the other hand, cannot interfere like this; they travel through slits one at a time. The figure below shows the pattern particles would make.
What would electrons do if they were also passed through two slits? Because we are used to thinking of electrons as individual particles, we may expect a “particle-like” interference pattern like this:

![Electrons and Film Diagram]

However, when this experiment is actually performed, we get quite an unexpected result – the electrons form an interference pattern similar to waves:

![Electrons and Film Diagram]

In fact, the interference pattern created by the electrons is identical to one created by light with a wavelength $\lambda = \frac{h}{p} \approx \frac{h}{mv}$ and with the same phase difference between the two slits. The approximation above is valid if $v \ll c$. $\lambda$ here is called the de Broglie wavelength. Just as light can exhibit particle-like properties (such as in the photoelectric effect), matter can exhibit wave like effects (such as interference patterns).
We learned when studying waves that interference patterns occur because the waves from each slit superpose with each other. What pattern would we expect to see if we sent through one electron, or for that matter one photon, at a time? As both the photon and the electron are supposed to be a single indivisible unit, and should go through one slit or another, we may expect to recover the “particle-like” interference pattern again.

For concreteness, let us discuss the experiment for photons – exactly the same analysis works for electrons as well. We will use a film to keep track of where the photons land, and let many photons through our slits, one at a time. The figure below shows the film after sending through a single photon (left) and after sending four photons (right).

So far it appears particle-like. However, as we continue sending photons through, we see that the pattern changes; the pattern begins to resemble that of a wave:
Somehow the individual photons going through the slits have managed to produce an interference pattern. Each photon makes one bright spot on the film (as in the photoelectric effect), so the light is still "particle-like," but it produces a pattern that we expect waves to make.

It seems like the particle (be it photon or electron) is traveling through both slits and interfering with itself! Because we usually consider photons and electrons as indivisible, this seems counter-intuitive. To get further insight on this, let us set up a detector by one of the slits that tells us which slit the photon travels through. That way we should be able to eliminate the possibility that the photon is “interfering with itself”. If we try this experiment with the detector, the result we obtain on our film is

Now that we can tell which photon went through which slit, we get the interference pattern we expect from particles. By switching the detector off and carrying out the experiment again, we recover the interference pattern for waves. When we tried to probe the particle nature of the light (e.g. which slit did it go through) we ended up with the pattern we expect particles to make. When we ignored which slit it went through, somehow the photon went through both and showed an interference pattern. In either case, how the light acts depends on what we are measuring! We can no longer simply ignore the influence of our measuring apparatus on the system, a subject we will address later.

There is another fundamental issue which has not been addressed. What distinguished different photons? What made one photon make a spot on the film, and another photon make a different spot? Nothing distinguishes these different photons. Unlike the marbles that were used to make the particle-like interference pattern, where marbles starting in different places ended in different places, photons are identical.

Given a single photon we cannot predict where it will land. However, we know it is much more likely to end up in a spot of constructive interference, and has no probability of ending up where there is complete destructive interference. We have given up deterministic physics – we must now settle with being able to calculate probabilities instead.

We have stated that this “matter wave” will give rise to an interference pattern. What other information can we retrieve from these tests? The interference pattern gives us a clue: the “brighter” the spot on the screen, the more likely it was for a particle to land there. We generalize this to a probability \(P\):

\[
P \left( \text{particle in region } x \pm \frac{\Delta x}{2} \right) \propto \text{ energy transmitted to that region}
\]

For light, we can find the energy transmitted to the area using \(I \Delta x\). This is true provided \((\Delta x)\) is small, so that the intensity is roughly constant over the region. If we want to know about a large region we would instead integrate the
intensity over that region. The intensity $I$ is proportional to the square of the field (wave amplitude). We then propose that we will get the right answer if:

$$\left[P \left( \text{particles in region } x \pm \frac{\Delta x}{2} \right) \propto | \psi (x,t)|^2 \Delta x \right]$$

where $| \psi (x,t) \rangle$ is the wave function for this matter wave. This is not a derivation, as quantum mechanics is a completely new phenomena that cannot be derived from our discussion of electromagnetism or Newton’s laws. Instead it is an argument to make the choice of $| \langle \psi |^2 \rangle$ seem less ad hoc. This choice has been verified by experiment to be taken seriously.

To summarize, the results of experiments are now only determined up to probabilities. If we plot $| \langle \psi (x,t)|^2 \rangle$ against $\langle x \rangle$, the probability of a particle being in a particular region is proportional to the area of that region. An example should clarify these points.

Example 10.3.1

Below is the “matter wave” for the electron at $(t=0)$. Which location or locations is the electron most likely to be found? Which location or locations is the electron least likely to be found?

Solution

We first want to turn this graph into a probability distribution, so we take the square of the graph. This gives us the graph below:
This graph peaks at \(x = 0\). Now the probability that the particle is at \textit{exactly} \(x = 0\) is \textit{zero} – the area under this point is tall but has basically zero width. However, the probability that the particle is \textit{around} \(x = 0\) is quite a bit higher than the probability it be around any other point. Compare the area under the graph around \(x = 0\) to the area around \(x = 2\). We see the particle is almost twice as likely to be close to \(x = 0\) than to \(x = 2\), and certainly more likely to be around either of those than around \(x = 3\). The graph below uses shaded rectangles to represent this concept.

Consider this, there is more area under the curve \textit{between} \(x = 0.5\) and \(x = 2.5\) than \textit{near} \(x = 0\). So it is typically more likely to find the particle between a range of \(x\) values than near any given \(x\) value. Taking this to the extreme, the
probability that the particle is somewhere must be 1, because it has to be somewhere. That means the region with the most probability is the one that extends from \((x = -\infty)\) to \((x = \infty)\)!

However, the question was asking which location was the particle most likely to be found, not which region was the most likely for the particle to be found. Choosing to focus only on tiny regions, we see the tiny around \((x = 0)\) it has the most area of any other tiny region we could pick, so we call \((x = 0)\) the most likely location for the particle. While the probability of being at any particular location is zero, the statement “the particle is most likely at \((x = 0)\)” is really a convenient shorthand for “the particle is more likely to be very close to \((x = 0)\) than to be very close to any other single point”. It is admittedly a sloppy use of language, but provided you know what is meant by such statements it will not lead you astray. The actual numerical value for the probability depends on how large we make our "tiny region".

The least likely locations for the particle are wherever the probability wave goes through zero. These locations are \((x = \pm 0.5, x = \pm 1.5, x = \pm 2.5)\) and pretty much anywhere with \(|x| > 3.5)\).

Matter waves are mostly analogous to the other waves we have seen so far in this course. One major difference is that matter waves have no polarization. The oscillations in a probability wave do not have a sensible interpretation as being in a direction so concept of longitudinal or transverse polarization simply does not apply.

More problematic is the issue of frequency and wave speed. When we discussed waves earlier, we related the frequency and wave speed with the equation \(v_{wave} = \lambda f\). To carry on the analogy with light, we also considered that \(E_{photon} = hf\). So far all of this is true, but we have glossed over one important detail: we do not know \(v_{wave}\) for a matter wave! In particular, it is not the same as the speed of the particle!

\[
\text{\[v_{wave}\neq v_{particle}\]}
\]

For light \((v_{\text{wave}} = v_{\text{photon}} = c)\). For matter waves, the wave velocity is much more complicated and at this level you should avoid thinking about the frequency of matter waves. Instead, we shall mean \(v_{\text{particle}}\) when we write \(v\), and restrict our attention to the wavelength \(\lambda = h/p\).

When we discussed harmonic oscillators, we used \(f\) in the formula for the energy levels. This \(f\) does not correspond to the frequency of the matter wave \(f_{wave}\), but refers to the frequency of the particle \(f_{particle}\) as it bounces back and forth. Because both the particle and the wave are oscillating this can cause some confusion. In this class the details are far too technical, and we will never look at the frequency of the matter wave. Whenever we write \(v\) or \(f\) in this section, we are always referring to \(v_{\text{particle}}\) or \(f_{\text{particle}}\), respectively.

In Quantized Energies, we presented the formulas for the allowed energy levels \(E_n\) in the infinite well, the simple harmonic oscillator and the hydrogen atom. Now we will show you how to actually find the energy levels. We will only explore the case of the infinite square well, because it is the simplest to do, but we indicate why the other cases are slightly harder. If you need a reminder on the infinite square well, return to our previous discussion.

Because the potential energy inside the well is zero, the total energy of the particle \(E\) is simply equal to the kinetic energy of the particle:
\[E=KE+0=\frac{1}{2}mv^2\]

The total energy is not changing, so the speed cannot change, and the magnitude of the momentum (\(|p|=m|v|\)) cannot change either. Because the momentum and wavelength are related by

\[
|\lambda| = \frac{h}{p} = \frac{h}{mv}
\]

we see that the matter wave has the same wavelength throughout the well. We also know that the particle cannot escape the box, so there is zero probability the particle is located outside the box. This tells us that the matter wave had better be zero outside the box. Because the matter wave should not undergo sudden jumps, the matter wave is restricted to certain wavelengths. Here are three examples of matter waves that have zero amplitude outside the box, and constant wavelength inside:

![Matter Waves Examples](image)

The first and last example work well, but in the second example the matter wave is discontinuous. The wavelength corresponding to this second example is not allowed in this system. The only wavelengths that are allowed are those that go to zero on the two walls, because the matter wave must be continuous. This is exactly the same as a standing wave with both of ends fixed, which we discussed in Standing Waves. The allowed wavelengths are then

\[|\lambda_n| = \frac{2L}{n}\]

where \(n=1,2,3,...\) is the number of anti-nodes in the wave. Recall that an anti-node is where the wave reaches a maximum or minimum; the left figure has one anti-node and the right one has two.

The momentum is related to the wavelength, so restricting the wavelengths to certain values also restricts the allowed values of momentum: \(|p_n|=\frac{h}{\lambda_n}=n\frac{h}{2L}\). We know that \(|p|=mv\), and because the mass does not change, we see that quantizing momentum also has the effect of quantizing velocities:

\[v_n= \frac{p_n}{m} = n\frac{h}{2Lm}\]

Finally, because we are only allowed certain values of the velocity we can only have certain values for the kinetic energy:

\[E_n = \frac{1}{2} mv_n^2=n^2\frac{h^2}{8 mL^2} =n^1E_1\]

which is exactly the formula we had before. Now we have some idea of what \(n\) is and how it got there: \(n\) is counting the number of anti-nodes in our matter wave!

Let us summarize the process:

- Standing probability waves \(\implies\) quantized \(|\lambda|\)
- Quantized \(|\lambda|\) \(\implies\) quantized \(|p|\)
Other potentials are more difficult to calculate exactly. One reason for this is the potential energy keeps changing, so the kinetic energy (and therefore wavelength) keep changing. However, with some more mathematical and physical consideration, it is possible to calculate the spectrum \(E_n\) of other potentials.

Recall how attempting to observe particle behavior in the double-slit experiment affected the experimental results. In that situation we discovered that trying to measure which path the particle took resulted in the interference pattern disappearing, and the particle-like pattern reappearing. This revealed that a measurement of a particle *inevitably* affects the particle being observed.

Measurement affects physical systems in more general ways. One very famous example is the **Heisenberg uncertainty principle** which states that you cannot know both the position and momentum of particle arbitrarily well. If you know the particle is in a region of size \(\sigma_x\), and you know the momentum is within some region \(\sigma_p\) then the following inequality must be satisfied

\[
\sigma_x \sigma_p \geq \frac{h}{4 \pi}
\]

Technically, \(\sigma_p\) and \(\sigma_x\) are the *standard deviations* of the measurements, but for most circumstances, treating them as the region we "absolutely know" the values is a sufficient consideration.

As we try to get a more accurate measurement of position (as \(\sigma_x \rightarrow 0\)) the information we have about the momentum gets worse (\(\sigma_p \geq \frac{h}{4 \pi \sigma_x}\)).

To see why something like this should be true, consider the act of measuring the position of a particle. One way of doing this would be to shine light on it, and see where it reflects (i.e. trace back the rays as we did for lenses). This is similar to how we see objects in everyday life:

We cannot determine the position of the particle to a size smaller than the wavelength of light we use, so the error involved in the measurement of the particles position is \(\sigma_x \approx \lambda\). As we use light of shorter and shorter wavelengths we get better accuracy on the particle’s position.

Light has momentum, so it imparts momentum on the particle as it changes direction:
Momentum is conserved, and the amount of momentum transferred to the particle depends on how much the light was deflected. If the light is only slightly deflected, the particle will still have (approximately) zero momentum. If the light bounces directly back, the momentum of the particle must be \( \text{twice} \) the initial momentum of the photon, \( \frac{2h}{\lambda} \). The uncertainty in the final momentum of the particle is roughly \( \sigma_p \approx \frac{1}{2} (p_{\text{max}} - p_{\text{min}}) = \frac{h}{\lambda} \). Putting these together we have

\[
\sigma_x \approx \frac{\hbar}{\lambda}
\]

The estimates presented above are rough, but this shows us that by using light of a long wavelength, we can measure momentum well \((p_{\text{light}} = 0)\) but know almost nothing about where the particle is. Likewise, we can sacrifice accuracy in our knowledge of momentum for more accuracy in position.

An obvious objection to the above argument is that there may be other ways of measuring position and momentum that do not involve light, we've only demonstrated that this particular method cannot accurately measure both simultaneously. This is a valid objection, but it turns out that \( \text{any} \) measuring procedure will disturb the system in such a way that you cannot simultaneously determine position and momentum completely.

### Hidden Variables

A more fundamental flaw with this derivation is that it pretends the particle is really a “marble-like” particle, and the uncertainty in position or momentum comes solely from our inability to measure it. This posits that there are things hidden away that we have not been clever enough to measure, collectively called “hidden variables”.

Quantum mechanics actually makes a much stronger statement: the uncertainty in measurements does not reflect our inability to make certain measurements; rather the particle does not really have a particular position or momentum until one makes a measurement. This may seem like a philosophical distinction; how could one show that these quantum particles were not really “marble-like,” that all the uncertainty came from our inability to measure? On the other side if we cannot measure it, can we really talk about it being true? There are experimental differences between “hidden variables” where the uncertainties are just from bad measurements, and quantum mechanics where the uncertainty is an inextricable part of the system. In all the experimental tests the predictions of quantum mechanics have been borne out.

What is the take home message from this? The argument above, using light to measure the position of a particle and demonstrate why the uncertainty principle is true, is one that appeals to our sense of how the world should work at the level of everyday things we are used to. However, it suffers major logical flaws e.g. how are we sure this is the best measurement that
can be performed? Quantum mechanics makes a much more radical departure: it claims that the world is not like everyday items at all, and in retrospect, it is somewhat amusing that this argument above gets us so close to the right answer!

- Authors of Phys7C (UC Davis Physics Department)