6.1: Phase and Group Velocity

Phase velocity is the speed at which a point of constant phase travels as the wave propagates.\(^1\) For a sinusoidally-varying wave, this speed is easy to quantify. To see this, consider the wave:

\[ A \cos(\omega t - \beta z + \psi) \]

where \(\omega = 2\pi f\) is angular frequency, \(z\) is position, and \(\beta\) is the phase propagation constant. At any given time, the distance between points of constant phase is one wavelength \(\lambda\). Therefore, the phase velocity \(v_p\) is

\[ v_p = \lambda f \]

Since \(\beta = \frac{2\pi}{\lambda}\), this may also be written as follows:

\[ v_p = \frac{\omega}{\beta} \]

Noting that \(\beta = \sqrt{\frac{\mu}{\epsilon}}\) for simple matter, we may also express \(v_p\) in terms of the constitutive parameters \(\mu\) and \(\epsilon\) as follows:

\[ v_p = \frac{1}{\sqrt{\mu\epsilon}} \]

Since \(v_p\) in this case depends only on the constitutive properties \(\mu\) and \(\epsilon\), it is reasonable to view phase velocity also as a property of matter.

Central to the concept of phase velocity is uniformity over space and time. Equations \ref{m0176_evp}-\ref{m0176_evpm} presume a wave having the form of Equation \ref{m0176_ewzt}, which exhibits precisely the same behavior over all possible time \(t\) from \(-\infty\) to \(+\infty\) and over all possible \(z\) from \(-\infty\) to \(+\infty\). This uniformity over all space...
and time precludes the use of such a wave to send information. To send information, the source of the wave needs to vary at least one parameter as a function of time; for example \(A\) (resulting in amplitude modulation), \(\omega\) (resulting in frequency modulation), or \(\psi\) (resulting in phase modulation). In other words, information can be transmitted only by making the wave non-uniform in some respect. Furthermore, some materials and structures can cause changes in \(\psi\) or other combinations of parameters which vary with position or time. Examples include dispersion and propagation within waveguides. Regardless of the cause, varying the parameters \(\omega\) or \(\psi\) as a function of time means that the instantaneous distance between points of constant phase may be very different from \(\lambda\). Thus, the **instantaneous** frequency of variation as a function of time and position may be very different from \(f\). In this case Equations \ref{m0176_evp}-\ref{m0176_evpm} may not necessarily provide a meaningful value for the speed of propagation.

Some other concept is required to describe the speed of propagation of such waves. That concept is **group velocity**, \(v_g\), defined as follows:

**Group velocity**, \(v_g\), is the ratio of the apparent change in frequency \(\omega\) to the associated change in the phase propagation constant \(\beta\); i.e., \((\Delta \omega/\Delta \beta)\).

Letting \(\Delta \beta\) become vanishingly small, we obtain

\[
\boxed{ v_g \triangleq \frac{\partial \omega}{\partial \beta} } \tag{m0176_evg}
\]

Note the similarity to the definition of phase velocity in Equation \ref{m0176_evp2}. Group velocity can be interpreted as the speed at which a disturbance in the wave propagates. Information may be conveyed as meaningful disturbances relative to a steady-state condition, so group velocity is also the speed of information in a wave.

Note Equation \ref{m0176_evg} yields the expected result for waves in the form of Equation \ref{m0176_ewzt}:

\[
\begin{align}
  v_g &= \left(\frac{\partial \beta}{\partial \omega}\right)^{-1} = \left(\frac{\partial}{\partial \omega} \omega \sqrt{\mu \epsilon}\right)^{-1} = \frac{1}{\sqrt{\mu \epsilon}} = v_p
\end{align}
\]

In other words, the group velocity of a wave in the form of Equation \ref{m0176_ewzt} is equal to its phase velocity.

To observe a difference between \(v_p\) and \(v_g\), \(\beta\) must somehow vary as a function of something other than just \(\omega\) and the constitutive parameters. Again, modulation (introduced by the source of the wave) and dispersion (frequency-dependent constitutive parameters) are examples in which \(v_g\) is not necessarily equal to \(v_p\). Here’s an example involving dispersion:

**Example \PageIndex{1}:** Phase and group velocity for a material exhibiting square-law dispersion

A broad class of non-magnetic dispersive media exhibit relative permittivity \(\epsilon_r\) that varies as the square of frequency over a narrow range of frequencies centered at \(\omega_0\). For these media we presume

\[
\epsilon_r = K \left(\frac{\omega}{\omega_0}\right)^2
\]

where \(K\) is a real-valued positive constant. What is the phase and group velocity for a sinusoidally-varying wave in this
material?

**Solution**

First, note

\[
\beta = \omega \sqrt{\mu_0 \epsilon} = \omega \sqrt{\mu_0 \epsilon_0} \sqrt{\epsilon_r} \\
= \frac{\sqrt{K} \cdot \omega^2}{\omega_0} \sqrt{\mu_0 \epsilon_0}
\]

The phase velocity is:

\[
v_p = \frac{\omega}{\beta} = \frac{\omega_0}{\sqrt{K} \cdot \omega \sqrt{\mu_0 \epsilon_0}}
\]

Whereas the group velocity is:

\[
v_g = \frac{\partial \omega}{\partial \beta} = \left( \frac{\partial \beta}{\partial \omega} \right)^{-1}
\]

\[
= \left( 2 \frac{\beta}{\omega} \right)^{-1} 
\]

Now simplifying using Equation \ref{m0176_eex1beta}:

\[
v_g = \left( 2 \frac{\omega}{\beta} \right)^{-1} = \frac{1}{2} v_p
\]

Thus, we see that in this case the group velocity is always half the phase velocity.

Another commonly-encountered example for which \(v_g\) is not necessarily equal to \(v_p\) is the propagation of guided waves; e.g., waves within a waveguide. In fact, such waves may exhibit phase velocity greater than the speed of light in a vacuum, \(c\). However, the group velocity remains less than \(c\), which means the speed at which information may propagate in a waveguide is less than \(c\). No physical laws are violated, since the universal “speed limit” \(c\) applies to information, and not simply points of constant phase. (See “Additional Reading” at the end of this section for more on this concept.)

1. Formally, “velocity” is a vector which indicates both the direction and rate of motion. It is common practice to use the terms “phase velocity” and “group velocity” even though we are actually referring merely to rate of motion. The direction is, of course, in the direction of propagation. $\triangleleft$

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