3: Relativity

Special relativity

The Lorentz transformation

The Lorentz transform \( (\vec{x}',t') = (\vec{x}',(\vec{x},t),t'(\vec{x},t)) \) leaves the wave equation invariant if \( c \) is invariant:

\[
\frac{\partial^2 }{\partial x^2} + \frac{\partial^2 }{\partial y^2} + \frac{\partial^2 }{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 }{\partial t^2} = \frac{\partial^2 }{\partial x'^2} + \frac{\partial^2 }{\partial y'^2} + \frac{\partial^2 }{\partial z'^2} - \frac{1}{c^2} \frac{\partial^2 }{\partial t'^2}
\]

This transform can also be found when \( ds^2 = ds'^2 \) is demanded. The general form of the Lorentz transform is given by:

\[
\vec{x}' = \vec{x} + \frac{(\gamma - 1)(\vec{x} \cdot \vec{v}') \vec{v}}{|v|^2} - \gamma \vec{v} t', \quad t' = \gamma \left( t - \frac{\vec{x} \cdot \vec{v}}{c^2} \right)
\]

where

\[
\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}
\]

The velocity difference \( \vec{v}' \) between two observers transforms according to:

\[
\vec{v}' = \left( \gamma \left(1 - \frac{\vec{v}_1 \cdot \vec{v}_2}{c^2} \right) \right)^{-1}
\]
If the velocity is parallel to the \((x)\)-axis, this becomes \(y'=y\) and \(z'=z\) and:

\[
\begin{aligned}
&x'=(\gamma(x-vt)~),~~~x=(\gamma(x'+vt')
&t'=(\gamma\left(t-\frac{xv}{c^2}\right)~),~~~t=(\gamma\left(t'+\frac{x'v}{c^2}\right)~),~~~v'=(\frac{v_2-v_1}{1-\frac{v_1v_2}{c^2}})
\end{aligned}
\]

If \(\vec{v}=(\gamma v\vec{e}_x)\):

\[
\begin{aligned}
&p'_{x}=\gamma(p_{x}-\frac{\beta W}{c})~,~~~W'=(\gamma(W-vp_{x})
\end{aligned}
\]

The electromagnetic field transforms according to:

\[
\begin{aligned}
&\vec{E}'=(\gamma(\vec{E}+\vec{v}\times\vec{B})~),~~~\vec{B}'=(\gamma(\vec{B}-\frac{\vec{v}\times\vec{E}}{c^2})
\end{aligned}
\]

Length, mass and time transform according to: \(\Delta t_{r}=(\gamma\Delta t_{0}\), \(m_{r}=(\gamma m_{0}\), \(l_{r}=(l_{0}/\gamma\), with \(\gamma\) labeling the quantities in a reference frame moving parallel at the same velocity and \((\{lm\})\) labeling the quantities in a frame moving with velocity \((v)\) w.r.t. it. The proper time \((\tau)\) is defined as: \((d\tau^2=ds^2/c^2)\), so \((\Delta\tau=\Delta t/\gamma)\). Energy and momentum are: \((W=m_{r}c^2=(\gamma W_{0}\), \((W_0^2=m_{0}^2c^4+p^2c^2)\). \((p=m_{r}v=(\gamma m_{0}v=Wv/c^2)\), and \((pc=W/\gamma)\) where \((\gamma=v/c)\). The force is defined by \((\vec{F}=d\vec{p}/dt)\).

4-vectors have the property that their modulus is independent of the observer: their components can change after a coordinate transform but not their modulus. The difference of two 4-vectors transforms also as a 4-vector. The 4-vector for the velocity is given by \((\vec{U}^\alpha=(\gamma u^i,ic\gamma)\). For particles with non-zero rest mass: \((U^\alpha U_\alpha=-c^2)\), for particles with zero rest mass (so with \((v=c)\)) then \((U^\alpha U_\alpha=0)\). The 4-vector for energy and momentum is given by:

\[
(p^\alpha=m_{0}U^\alpha=(\gamma(p^i,iW/c))\). So: \((p_\alpha p^\alpha=-m_0^2c^2=p^2-W^2/c^2)\).

Red and blue shift

There are three causes of red and blue shifts:

1. Motion: with \((\vec{e}_x\cdot\vec{e}_r=\cos(\varphi))\) from which follows: \((\vec{dx}/d\tau=\gamma\vec{e}_x)\). The relation with the “common” velocity \((\vec{u}=dx/dt)\) is: \((U^\alpha=\gamma(u^i,ic\gamma)\). For particles with non-zero rest mass: \((U^\alpha U_\alpha=-c^2)\), for particles with zero rest mass (so with \((v=c)\)) then \((U^\alpha U_\alpha=0)\). The 4-vector for energy and momentum is given by:

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The stress-energy tensor and the field tensor

The stress-energy tensor is given by:

\[
T_{\mu\nu}=(\rho c^2+p)u_\mu u_\nu+p g_{\mu\nu}+\frac{1}{c^2} \left(F_{\mu\alpha}F^{\alpha\nu}+\frac{1}{4}g_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right)
\]

The conservation laws can than be written as: \(\nabla_\nu T^{\mu\nu}=0\). The electromagnetic field tensor is given by:

\[
F_{\alpha\beta}=\partial_\alpha A_\beta-\partial_\beta A_\alpha
\]

with \(A_\mu= (\vec{A},iV/c)\) and \(J_\mu= (\vec{J},ic\rho)\). Maxwell’s equations can then be written as:

\[
\partial_\nu F^{\mu\nu}=\mu_0 J^\mu,~~ \partial_\lambda F_{\mu\nu}+\partial_\mu F_{\nu\lambda}+\partial_\nu F_{\lambda\mu}=0
\]

The equations of motion for a charged particle in an EM field become with the field tensor:

\[
\frac{dp_\alpha}{d\tau}=qF_{\alpha\beta}u^\beta
\]

General relativity

Riemann geometry, the Einstein tensor

The basic principles of general relativity are:

1. The geodesic postulate: free falling particles move along geodesics of space-time with the proper time \(\tau\) or arc length \(s\) as parameter. For particles with zero rest mass (photons), the use of a free parameter is required because for them \(ds=0\). From \(\Delta s=0\) the equations of motion can be derived:

\[
\frac{d^2x^\alpha}{ds^2}+\Gamma_{\beta\gamma}^\alpha \frac{dx^\beta}{ds} \frac{dx^\gamma}{ds}=0
\]

2. The principle of equivalence: inertial mass \(\equiv\) gravitational mass \(\Rightarrow\) gravitation is equivalent with a curved space-time where particles move along geodesics.

3. By a proper choice of the coordinate system it is possible to make the metric locally flat in each point \((x_i)\):

\[
g_{\alpha\beta}(x_i)=\eta_{\alpha\beta}:=\text{diag}((-1,1,1,1))
\]

The Riemann tensor is defined as:

\[
R^\mu_{\nu\alpha\beta}T^\nu:=\nabla_\alpha\nabla_\beta T^\mu_{\nu}\nabla_\nu\nabla_\mu,\text{ where the covariant derivative is given by:} \nabla_j a^i=\partial_j a^i+\Gamma_{jk}^i a^k \text{ and} \nabla_j a_i=\partial_j a_i-\Gamma_{ij}^k a_k. \text{ Here,}
\]

\[
\Gamma_{jk}^i=\frac{g^{il}}{2}\left(\frac{\partial g_{lj}}{\partial x^k}+\frac{\partial g_{lk}}{\partial x^j}-\frac{\partial g_{kl}}{\partial x^j}\right)
\]
for Euclidean spaces this reduces to:

$$\Gamma_{jk}^i = \frac{\partial^2 \bar{x}^l}{\partial x^j \partial x^k} \frac{\partial x^i}{\partial \bar{x}^l}.$$ 

where $$\Gamma_{jk}^i$$ are the Christoffel symbols. For a second-order tensor

$$\langle \nabla_\alpha \nabla_\beta T^\mu_\nu \rangle = \Gamma^\mu_\sigma \partial_\alpha T^\sigma_\nu + \Gamma^\sigma_\nu \partial_\beta T^\mu_\sigma + R_{\nu\alpha \beta}^\mu T^\sigma_\nu + R_{\nu\sigma \beta}^\mu T^\mu_\nu,$$

$$\nabla_\alpha a^i_j = \partial_\alpha a^i_j - \Gamma^i_{kj} a^k_l + \Gamma^k_{jl} a^i_k,$$

$$\nabla_\alpha a_{ij} = \partial_\alpha a_{ij} - \Gamma^i_{ki} a^l_j - \Gamma^j_{kj} a^i_l,$$

$$\nabla_\alpha a^{ij} = \partial_\alpha a^{ij} + \Gamma^i_{kl} a^{lj} + \Gamma^j_{kl} a^{il}.$$ The following holds: $$\langle R_{\nu\alpha \beta}^\mu \rangle \ = \partial_\mu \Gamma_{\nu\beta}^\alpha - \partial_\nu \Gamma_{\nu\alpha}^\beta + \Gamma_{\sigma\nu}^\alpha \Gamma_{\nu\beta}^\sigma - \Gamma_{\nu\alpha}^\beta \Gamma_{\sigma\nu}^\sigma.$$ The Ricci tensor is a contraction of the Riemann tensor: $$\langle R_{\nu\alpha} \rangle = \langle R^\mu_{\alpha\mu\nu} \rangle,$$ which is symmetric: $$\langle R_{\nu\alpha} \rangle = \langle R_{\alpha\nu} \rangle.$$ The Bianchi identities are: $$\langle \nabla_\lambda R_{\nu\alpha\beta\mu} \rangle + \langle \nabla_\nu R_{\alpha\beta\lambda\mu} \rangle + \langle \nabla_\mu R_{\alpha\beta\nu\lambda} \rangle = 0.$$ The Einstein tensor is given by: $$G^\alpha_\beta := R^\alpha_\beta - \frac{1}{2} g^\alpha_\beta R,$$ where $$R := R^\mu_\alpha \eta^\mu_\mu.$$ The Laplace equation for Newtonian gravitation can be derived by starting with: $$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$ where $$|h| \ll 1.$$ In the stationary case, this results in $$\nabla^2 h_{00} = \frac{8\pi \kappa}{c^2} \rho.$$ The most general form of the field equations is:

$$\langle \nabla_\lambda R_{\nu\alpha\beta\mu} - \frac{1}{2} g_{\nu\alpha} R + \Lambda g_{\nu\alpha} \rangle = \frac{8\pi \kappa}{c^2} T_{\nu\alpha},$$

where $$\Lambda$$ is the cosmological constant. This constant plays a role in inflationary models of the universe.

The line element

The metric tensor in an Euclidean space is given by:

$$g_{\alpha\beta} = \sum_k \frac{\partial \bar{x}^k}{\partial x^\alpha} \frac{\partial \bar{x}^k}{\partial x^\beta}.$$ 

In general: $$\langle ds^2 = g_{\mu\nu} dx^\mu dx^\nu \rangle.$$ In special relativity this becomes $$\langle ds^2 = c^2 dt^2 + dx^2 + dy^2 + dz^2 \rangle.$$ This metric, $$\langle \eta_{\mu\nu} := \text{diag}((-1,1,1,1)) \rangle,$$ is called the Minkowski metric.
The external Schwarzschild metric applies in vacuum outside a spherical mass distribution, and is given by:

\[ds^2=\left(-1+\frac{2m}{r}\right)c^2dt^2+\left(1-\frac{2m}{r}\right)^{-1}dr^2+r^2d\Omega^2\]

Here, \(m:=M\kappa/c^2\) is the geometrical mass of an object with mass \(M\), and \((d\Omega^2=d\theta^2+\sin^2\theta d\varphi^2)\). This metric is singular for \(r=2m=2\kappa M/c^2\). If an object is smaller than its event horizon \((2m)\), that implies that its escape velocity is \(>c\), it is called a black hole. The Newtonian limit of this metric is given by:

\[ds^2=-(1+2V)c^2dt^2+(1-2V)(dx^2+dy^2+dz^2)\]

where \((V:=\kappa M/r)\) is the Newtonian gravitation potential. In general relativity, the components of \((g_{\mu\nu})\) are associated with the potentials and the derivatives of \((g_{\mu\nu})\) with the field strength.

The Kruskal-Szekeres coordinates are used to solve certain problems with the Schwarzschild metric near \((r=2m)\). They are defined by:

- \((r>2m)\):
  \[
  \left\{ \begin{array}{c}
  u &=& \sqrt{\frac{r}{2m}-1}\exp\left(\frac{r}{4m}\right)\cosh\left(\frac{t}{4m}\right) \\
  v &=& \sqrt{\frac{r}{2m}-1}\exp\left(\frac{r}{4m}\right)\sinh\left(\frac{t}{4m}\right)
  \end{array} \right. \\
  \end{aligned}
  \]

- \((r<2m)\):
  \[
  \left\{ \begin{array}{c}
  u &=& \sqrt{1-\frac{r}{2m}}\exp\left(\frac{r}{4m}\right)\sinh\left(\frac{t}{4m}\right) \\
  v &=& \sqrt{1-\frac{r}{2m}}\exp\left(\frac{r}{4m}\right)\cosh\left(\frac{t}{4m}\right)
  \end{array} \right. \\
  \end{aligned}
  \]

- \((r=2m)\): here, the Kruskal coordinates are singular, which is necessary to eliminate the coordinate singularity there.

The line element in these coordinates is given by:

\[ds^2=-\frac{32m^3}{r}\exp^{-r/2m}(dv^2-du^2)+r^2d\Omega^2\]

The line \((r=2m)\) corresponds to \((u=v=0)\), the limit \((x^0\rightarrow\infty)\) with \((u=\nu)\) and \((x^0\rightarrow-\infty)\) with \((u=-\nu)\). The Kruskal coordinates are only singular on the hyperbola \((v^2-u^2=1)\), this corresponds with \((r=0)\). On the line \((dv=\pm du)\) \((d\theta=d\varphi=ds=0)\) holds.

For the metric outside a rotating, charged spherical mass the Newman metric applies:

\[
\begin{aligned}
\begin{aligned}
&ds^2=-(1-\frac{2m}{r})c^2dt^2+(1-\frac{2m}{r})^{-1}dr^2+r^2d\Omega^2-\frac{2mr-e^2}{r^2+a^2\cos^2\theta}dr^2-\frac{r^2+a^2\cos^2\theta}{r^2-2mr+a^2-e^2}d\theta^2-rac{(2mr-e^2)a^2\sin^2\theta}{r^2+a^2\cos^2\theta}d\varphi^2-\frac{2a(2mr-e^2)}{r^2+a^2\cos^2\theta}d\varphi dt
\end{aligned}
\end{aligned}
\]

where \((m:=\kappa M/c^2)\), \((a=L/Mc)\) and \((e:=\kappa Q/\varepsilon_0c^2)\).

A rotating charged black hole has an event horizon with \((R_{\text{E}}=m+\sqrt{m^2-a^2\cos^2\theta})\).

Near rotating black holes frame dragging occurs because \((g_{\mu\nu}\varphi \neq 0)\). For the Kerr metric \((e=0)\), \((a\neq 0)\) then follows that within the surface \((R_{\text{E}}=m+\sqrt{m^2-a^2\cos^2\theta})\) (the ergosphere) no particle can be at rest.
Planetary orbits and the perihelion shift

To find a planetary orbit, the variational problem \(\delta\int ds=0\) has to be solved. This is equivalent to the problem \(\delta\int ds^2=\delta\int g_{ij}\partial x^i\partial x^j=0\). Substituting the external Schwarzschild metric yields for a planetary orbit:

\[
\frac{du}{d\varphi}\left(\frac{d^2u}{d\varphi^2}+u\right)=\frac{du}{d\varphi}\left(3mu+\frac{m}{h^2}\right)
\]

where \(u:=1/r\) and \(h=r^2\dot{\varphi}\) constant. The term \((3mu)\) is not present in the classical solution. This term can also be found in the classical case from a potential \(V(r)=-\frac{\kappa M}{r}(1+\frac{h^2}{r^2})\).

The orbital equation gives \(r=\)constant as solution, or can, after dividing by \(\frac{du}{d\varphi}\), be solved with perturbation theory. In zeroth order, this results in an elliptical orbit: \(u_0(\varphi)=A+B\cos(\varphi)\) with \((A=m/h^2)\) and \((B)\) an arbitrary constant. In first order, this becomes:

\[
[u_1(\varphi)=A+B\cos(\varphi-\epsilon\varphi)+\epsilon(A+\frac{B^2}{2A}-\frac{B^2}{6A}\cos(2\varphi))
\]

where \((varepsilon=3m^2/h^2)\) is small. The perihelion of a planet is the point for which \(u=0\) a minimum or \(u=0\) a maximum. This is the case if \(\cos(\varphi-\epsilon\varphi)=0\Rightarrow\varphi\approx2\pi n(1+\epsilon)\). For the perihelion shift it then follows that: \(\Delta\varphi=2\pi\epsilon=6\pi m^2/h^2\) per orbit.

The trajectory of a photon

For the trajectory of a photon (and for each particle with zero rest mass) \(ds^2=0\). Substituting the external Schwarzschild metric results in the following orbital equation:

\[
\frac{du}{d\varphi}\left(\frac{d^2u}{d\varphi^2}-3mu\right)=0
\]

Gravitational waves

Starting with the approximation \(g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}\) for weak gravitational fields and the definition \(h_{\mu\nu}=h_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}h_{\alpha\alpha}\) it follows that \(\Box h_{\mu\nu}=0\) if the gauge condition \(\partial h_{\mu\nu}/\partial x^\nu=0\) is satisfied. From this, it follows that if the relative velocities are \((\ll c)\) and the wavelengths \((gg)\) than the size of the system, the loss of energy of a mechanical system is given by:

\[
\frac{dE}{dt}=-\frac{G}{5c^5}\sum_{ij}\left(\frac{d^3Q_{ij}}{dt^3}\right)^2
\]

with \((Q_{ij}=\int\varrho(x_{ixj}-\frac{1}{3}\delta_{ij}r^2)d^3x)\) the mass quadrupole moment.
Cosmology

If for the universe as a whole it is assumed:

1. There exists a global time coordinate which acts as \(x^0\) of a Gaussian coordinate system,
2. The 3-dimensional spaces are isotrope for a certain value of \(x^0\),
3. Each point is equivalent to each other point for a fixed \(x^0\).

then the Robertson-Walker metric can be derived for the line element:

\[
ds^2 = -c^2dt^2 + \frac{R^2(t)}{r_0^2\left(1-\frac{kr^2}{4r_0^2}\right)}(dr^2 + r^2d\Omega^2)\]

For the scale factor \(R(t)\) the following equations can be derived:

\[
\frac{2\ddot{R}}{R} + \frac{\dot{R}^2 + kc^2}{R^2} = -\frac{8\pi\kappa p}{c^2} + \Lambda \quad \text{and} \quad \frac{\dot{R}^2 + kc^2}{R^2} = \frac{8\pi\kappa\varrho}{3} + \frac{\Lambda}{3}
\]

where \(\rho\) is the pressure and \(\varrho\) the density of the universe. If \(\Lambda = 0\) the deceleration parameter \(q\) can be derived:

\[
q = -\frac{\ddot{R}}{\dot{R}^2} = \frac{4\pi\kappa\varrho}{3H^2}
\]

where \(H = \frac{\dot{R}}{R}\) is Hubble’s constant. This is a measure of the velocity with which galaxies far away are moving away from each other, and has the value \(\approx(75 \pm 25)\) km\(\cdot\)s\(^{-1}\)\(\cdot\)Mpc\(^{-1}\). This gives three possible conditions for the universe (here, \(W\) is the total amount of energy in the universe):

1. **Parabolical universe**: \(k = 0\), \(W = 0\), \(q = \frac{\dot{R}}{\dot{R}^2}\). The expansion velocity of the universe \(\rightarrow 0\) if \(t \rightarrow \infty\). The associated critical density is \(\varrho_c = 3H^2/8\pi\kappa\).
2. **Hyperbolical universe**: \(k = -1\), \(W < 0\), \(< q < \frac{\dot{R}}{\dot{R}^2}\). The expansion velocity of the universe remains positive forever.
3. **Elliptical universe**: \(k = 1\), \(W > 0\), \(q > \frac{\dot{R}}{\dot{R}^2}\). The expansion velocity of the universe becomes negative after some time: the universe starts collapsing.