**15: Quantum Field Theory and Particle Physics**

**Creation and annihilation operators**

A state with many particles can be described by collective occupation numbers \(|n_1n_2n_3\cdotsrangle\). Hence the vacuum state is given by \(|000\cdotsrangle\). This is a complete description because the particles are indistinguishable. The states are orthonormal:

\[
\langle n_1n_2n_3\cdots|n_1'n_2'n_3'\cdots\rangle=\prod_{i=1}^{\infty}\delta_{n_in_i'}
\]

The time-dependent state vector is given by

\[
|\Psi(t)\rangle=\sum_{n_1n_2\cdots}c_{n_1n_2\cdots}(t)|n_1n_2\cdotsrangle
\]

The coefficients \(c\) can be interpreted as follows: \(|(c_{n_1n_2\cdots}|^2\rangle\) is the probability to find \(|n_1\rangle\) particles with momentum \(|\text{vec}\{k\}_1\rangle\), \(|n_2\rangle\) particles with momentum \(|\text{vec}\{k\}_2\rangle\), etc., and \(|\langle\Psi(t)|\Psi(t)|\rangle\rangle=\sum|c_{n_i}(t)|^2=1\). The expansion of the states in time is described by the Schrödinger equation

\[
i\frac{d}{dt}|\Psi(t)\rangle=H|\Psi(t)\rangle
\]

where \(|H=H_0+H_{\text{int}}\rangle\). \(|H_0\rangle\) is the Hamiltonian for free particles and keeps \(|(c_{n_i}|(t)|^2\rangle\) constant, \(|H_{\text{int}}\rangle\) is the interaction Hamiltonian and can increase or decrease a \(|(c^\dagger2\rangle\) at the cost of others.

All operators which can change occupation numbers can be expanded in the \(|a\rangle\) and \(|a^\dagger\rangle\) operators. \(|a\rangle\) is the **annihilation operator** and \(|a^\dagger\rangle\) the **creation operator**, and:
\[
\begin{aligned}
 a(\vec{k}_i)|n_1n_2\cdots n_i\cdots\rangle &= \sqrt{n_i}|n_1n_2\cdots n_i-1\cdots\rangle \\
 a^\dagger(\vec{k}_i)|n_1n_2\cdots n_i\cdots\rangle &= \sqrt{n_i+1}|n_1n_2\cdots n_i+1\cdots\rangle
\end{aligned}
\]

Because the states are normalized \(\langle a|0\rangle=0\) and \(\langle a(\vec{k}_i)a^\dagger(\vec{k}_i)|n_i\rangle=n_i\langle n_i\rangle\). So \(\langle aa^\dagger\rangle\) is an occupation number operator. The following commutation rules can be derived:

\[
[a(\vec{k}_i),a(\vec{k}_j)]=0,~~\ldots\ldots [a^\dagger(\vec{k}_i),a^\dagger(\vec{k}_j)]=0,~~\ldots\ldots
[a(\vec{k}_j),a^\dagger(\vec{k}_i)]=\delta_{ij}]
\]

Hence for free spin-0 particles: \(\langle H_0=\sum\limits_i a^\dagger(\vec{k}_i)a(\vec{k}_i)\hbar\omega_{k_i}\rangle\)

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**Classical and quantum fields**

Starting with a real field \(\langle\Phi^\alpha(x)\rangle\) (complex fields can be split into a real and an imaginary part), the *Lagrange density* \(\langle\text{cal L}\rangle\) is a function of the position \(\langle x=(\vec{x},ict)\rangle\) through the fields: \(\langle\text{cal L}\rangle=\text{cal L}(\langle\Phi^\alpha(x),\partial_\nu\Phi^\alpha(x)\rangle)\). The *Lagrangian* is given by \(\langle L=\int\text{cal L}(x)d^3x\rangle\). Using the variational principle \(\langle\delta I(\Omega)=0\rangle\) and with the action-integral \(\langle I(\Omega)=\int\text{cal L}(\langle\Phi^\alpha,\partial_\nu\Phi^\alpha\rangle)d^4x\rangle\) the field equation can be derived:

\[
\frac{\partial \text{cal L}}{\partial \Phi^\alpha}-\frac{\partial }{\partial x_\nu}\frac{\partial \text{cal L}}{\partial (\partial_\nu\Phi^\alpha)}=0
\]

The *conjugate field* is, analogous to momentum in classical mechanics, defined as:

\[
\Pi^\alpha(x)=\frac{\partial \text{cal L}}{\partial \dot{\Phi}^\alpha}
\]

With this, the *Hamilton* density becomes \(\langle\{\text{cal H}(x)=\Pi^\alpha\dot{\Phi}^\alpha-\text{cal L}(x)\}\rangle\). Quantization of a classical field is analogous to quantization in point mass mechanics: the field functions are considered as operators obeying certain commutation rules:

\[
[\langle\Phi^\alpha(\vec{x}),\Phi^\beta(\vec{x}\,')\rangle]=0,~~\ldots\ldots
[\langle\Pi^\alpha(\vec{x}),\Pi^\beta(\vec{x}\,')\rangle]=0,~~\ldots\ldots[\langle\Phi^\alpha(\vec{x}),\Pi^\beta(\vec{x}\,')\rangle]=i\delta_{\alpha\beta}(\langle\vec{x}\,-\vec{x}\,'\rangle)
\]

---

**The interaction picture**

Some equivalent formulations of quantum mechanics are possible:

1. *Schrödinger picture*: time-dependent states, time-independent operators.
The interaction picture can be obtained from the Schrödinger picture by an unitary transform:

\[
|\Phi(t)\rangle = e^{iH_0^S} |\Phi^S(t)\rangle \quad \text{and} \quad O(t) = e^{iH_0^S} O^S e^{-iH_0^S} \]

The index \(^\text{S}\) denotes the Schrödinger picture. From this follows:

\[
\frac{i}{\text{d}t} |\Phi(t)\rangle = H_{\text{int}}(t) |\Phi(t)\rangle \quad \text{and} \quad \frac{i}{\text{d}t} O(t) = \{O(t), H_0\} \]

Real scalar field in the interaction picture

It is easy to find that, with \(M := m_0^2 c^2 / \hbar^2\), it holds that:

\[
\frac{\partial}{\partial t} \Phi(x) = \Pi(x) \quad \text{and} \quad \frac{\partial}{\partial t} \Pi(x) = (\nabla^2 - M^2) \Phi(x) \]

From this follows that \(|\Phi\rangle\) obeys the Klein-Gordon equation \((\Box - M^2) \Phi = 0\). With the definition \(k_0^2 = \vec{k}^2 + M^2 = \omega_k^2\) and the notation \(\langle \vec{k} \mid \text{c} \mid \vec{x}, \text{t} \rangle := kx\) the general solution of this equation is:

\[
|\Phi(x)\rangle = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \frac{1}{\sqrt{2\omega_k}} \left( a(\vec{k}) e^{ikx} + a^\dagger(\vec{k}) e^{-ikx} \right), \quad \Pi(x) = \frac{i}{\sqrt{V}} \sum_{\vec{k}} \sqrt{\frac{1}{2}\omega_k} \left( -a(\vec{k}) e^{ikx} + a^\dagger(\vec{k}) e^{-ikx} \right) \]

The field operators contain a volume \(V\), which is used as normalization factor. Usually one can take the limit \(V \rightarrow \infty\).

In general it holds that the term with \(|e^{-ikx}\rangle\), the positive frequency part indicates creation, and the negative frequency indicates annihilation.

The coefficients have to be each others Hermitian conjugate because \(|\Phi\rangle\) is Hermitian. Because \(|\Phi\rangle\) has only one component this can be interpreted as a field describing a particle with spin zero. From this follows that the commutation rules are given by \(|\langle \Phi(x), \Phi(x') |\rangle = i \Delta(x-x')\) with

\[
|\Delta(y)\rangle = \frac{1}{(2\pi)^3} \int \frac{\sin(ky)}{\omega_k} \text{d}^3k \]

\(|\Delta(y)\rangle\) is an odd function which is invariant for proper Lorentz transforms (no mirroring). This is consistent with the previously found result \(|\langle \Phi(\vec{x}, t), \Phi(\vec{x'}, t') \rangle = 0\). In general it holds that \(|\Delta(y)\rangle = 0\) outside the light cone. So the equations obey the locality postulate.

The Lagrange density is given by: \(|\langle \Phi, \text{partial} / \text{nu} \Phi |\rangle = \frac{1}{2} \left( (\text{partial} / \text{nu}) \Phi + \text{m}^2 \Phi^2 \right)\). The energy operator is given by:

\[
[H = \int |\text{cal H}(x) \rangle \text{d}^3x = \text{sum} \{\vec{k}\} \hbar \omega_k \text{ka}^\dagger \text{dagger} \langle \vec{k} \mid a(\vec{k}) \} \}
\]
Charged spin-0 particles, conservation of charge

The *Lagrange density* of charged spin-0 particles is given by: 
\[
\mathcal{L} = -(\partial_\nu \Phi \partial_\nu \Phi^* + M^2 \Phi \Phi^*)
\]

*Noether’s theorem* connects a continuous symmetry of \(\mathcal{L}\) and an additive conservation law. Suppose that \(\mathcal{L}(\Phi', \partial_\nu \Phi') = \mathcal{L}(\Phi, \partial_\nu \Phi)\) and there exists a continuous transform between \(\Phi\) and \(\Phi'\) such as \(\Phi' = \Phi + \epsilon f(\Phi)\). Then 
\[
\frac{\partial }{\partial x_\nu}\left(\frac{\partial \mathcal{L}}{\partial (\partial_\nu \Phi)}f\right) = 0
\]
This is a continuity equation \(\nabla\text{Rightarrow}\) conservation law. Which quantity is conserved depends on the symmetry. The above Lagrange density is invariant for a change in phase \(\Phi \rightarrow \Phi e^{i\theta}\): a global gauge transform. The conserved quantity is the current density \(J_\mu(x) = -ie(\Phi \partial_\mu \Phi^* - \Phi^* \partial_\mu \Phi)\). Because this quantity is 0 for real fields a complex field is needed to describe charged particles. When this field is quantized the field operators are given by 
\[
\Phi(x) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \frac{1}{\sqrt{2\omega_k}} \left(a(\vec{k},) e^{i\vec{k} \cdot \vec{x}} + b(\vec{k},) e^{-i\vec{k} \cdot \vec{x}}\right) \quad \text{and} \quad \Phi^\dagger(x) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \frac{1}{\sqrt{2\omega_k}} \left(a^\dagger(\vec{k},) e^{i\vec{k} \cdot \vec{x}} + b^\dagger(\vec{k},) e^{-i\vec{k} \cdot \vec{x}}\right)
\]
Hence the energy operator is given by:
\[
H = \sum_{\vec{k}} \hbar \omega_{\vec{k}} \left(a^\dagger(a) + b^\dagger b\right)
\]
and the charge operator is given by:
\[
Q(t) = -i \int J_4(x) d^3x \Rightarrow Q = \sum_{\vec{k}} e \left(a^\dagger a - b^\dagger b\right)
\]
From this follows that \(a^\dagger a = N_+(\vec{k})\) is an occupation number operator for particles with a positive charge and \(b^\dagger b = N_-(\vec{k})\) is an occupation number operator for particles with a negative charge.

Field functions for spin-1/2 particles

Spin is defined by the behaviour of the solutions \(\psi\) of the *Dirac equation*. A scalar field \(\Phi(x)\) has the property that, if it obeys the *Klein-Gordon equation*, the rotated field \(\tilde{\Phi}(x) := \Phi(\Lambda^{-1}x)\) also obeys it. \(\Lambda\) denotes 4-dimensional rotations: the proper Lorentz transforms. These can be written as:
\[
[\tilde{\Phi}(x) = \Phi(x) \lnot \cdot \dot{e} + \epsilon \text{dot} \cdot \text{vec} \cdot \vec{L}] \
\]

For $0 \leq \mu \leq 3, \nu \leq 3$ these are rotations, for $\nu = 4, \mu \neq 4$ these are Lorentz transformations.

A rotated field $\tilde{\psi}(x)$ obeys the Dirac equation if the following condition holds:
$\tilde{\psi}(x) = D(\Lambda(x))\psi(\Lambda^{-1}(x))$. This results in the condition $D^{-1}\gamma_\lambda D = \Lambda_{\lambda\mu}\gamma_\mu$. One finds: $D = \frac{1}{2} \hbar \gamma_\mu \gamma_\nu S_{\mu\nu}$. Hence:

$$\tilde{\psi}(x) = \psi(x) e^{iS+L}$$

Then the solutions of the Dirac equation are given by:

$$\psi(x) = u_\pm r(\vec{p}) e^{-i(\vec{p} \cdot \vec{x} \mp Et)}$$

Here, $(r)$ indicates the direction of the spin, and $(\pm)$ is the sign of the energy. With the notation $u^r(\vec{p}) = u^r_-(\vec{p})$ and $u'^r(\vec{p}) = u'^r_+(\vec{p})$ one can write for the dot products of these spinors:

$$u^r_-(\vec{p}) u'^r_-(\vec{p}) = \frac{E}{M} \delta_{rr'}, u^r_+(\vec{p}) u'^r_+(\vec{p}) = \frac{E}{M} \delta_{rr'}, u^r_-(\vec{p}) u'^r_+(\vec{p}) = 0$$

Because of the factor $(E/M)$ this is not relativistically invariant. A Lorentz-invariant dot product is defined by $\overline{a} b := a^\dagger \gamma_4 b$, where $\overline{a} := a^\dagger \gamma_4$ is a row spinor. From this follows:

$$\overline{u^r(\vec{p})} u^{r'}(\vec{p}) = \delta_{rr'}, \overline{v^r(\vec{p})} v^{r'}(\vec{p}) = -\delta_{rr'}, \overline{u^r(\vec{p})} v^{r'}(\vec{p}) = 0$$

Combinations of the type $a \overline{a}$ give a $4 \times 4$ matrix:

$$\sum_{r=1}^2 u^r(\vec{p}) \overline{u^r(\vec{p})} = \frac{-i\gamma_\lambda p_\lambda + M}{2M}, \sum_{r=1}^2 v^r(\vec{p}) \overline{v^r(\vec{p})} = \frac{-i\gamma_\lambda p_\lambda - M}{2M}$$

The Lagrange density which results in the Dirac equation and has the correct energy normalization is:

$${\mathcal L}(x) = -\overline{\psi(x)} (\gamma_\mu \partial_\mu + M) \psi(x)$$

and the current density is $J_\mu(x) = -ie \overline{\psi}(x) \gamma_\mu \psi(x)$.

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**Quantization of spin-1/2 fields**

The general solution for the field operators is in this case:

$$\psi(x) = \sqrt{\frac{M}{V}} \sum_{\vec{p}} \frac{1}{\sqrt{E}} \sum_r \left[ c_r(\vec{p}) u^r(\vec{p}) e^{i\vec{p} \cdot \vec{x}} + d^r \dagger v^r(\vec{p}) e^{-i\vec{p} \cdot \vec{x}} \right]$$

and
\[
\psi(x) = \sqrt{\frac{M}{V}} \sum_{\vec{p}} \frac{1}{\sqrt{E}} \sum_r \left( c_r^\dagger(\vec{p}\,) \overline{u^r}(\vec{p}\,) e^{-ipx} + d_r(\vec{p}\,) \overline{v^r}(\vec{p}\,) e^{ipx} \right)
\]

Here, \(c^\dagger\) and \(c\) are respectively the creation and annihilation operators for an electron and \(d^\dagger\) and \(d\) the creation and annihilation operators for a positron. The energy operator is given by

\[
H = \sum_{\vec{p}} E_{\vec{p}} \sum_{r=1}^{2} \left( c_r^\dagger(\vec{p}\,) c_r(\vec{p}\,) - d_r(\vec{p}\,) d_r^\dagger(\vec{p}\,) \right)
\]

To prevent the energy of positrons being negative the operators must obey anti commutation rules instead of commutation rules:

\[
[c_r(\vec{p}\,), c_{r'}^\dagger(\vec{p}\,)]_+ = [d_r(\vec{p}\,), d_{r'}^\dagger(\vec{p}\,)]_+ = \delta_{rr'} \delta_{pp'} ,
\]

all other anti commutators are 0. The field operators obey

\[
[[\psi_\alpha(x), \psi_\beta(x)] = 0 , [\overline{\psi}_\alpha(x), \overline{\psi}_\beta(x)] = 0 , [\psi_\alpha(x), \overline{\psi}_\beta(x)]_+ = -iS_{\alpha\beta}(x-x')
\]

with

\[
S(x) = \left( \gamma_\lambda \frac{\partial}{\partial x_\lambda} - M \right) \Delta(x)
\]

Besides providing a positive-definite energy the anti commutation rules lead to the Pauli exclusion principle and Fermi-Dirac statistics: because \(c_r^\dagger(\vec{p}\,) c_r^\dagger(\vec{p}\,) = c_r^\dagger(\vec{p}\,) c_r^\dagger(\vec{p}\,)\) and thus:

\(\langle c_r^\dagger(\vec{p}\,)^2 \rangle = 0\). It appears to be impossible to create two electrons with the same momentum and spin. This is the exclusion principle. Another way to see this is the fact that \(\langle N^+_r(\vec{p}\,) \rangle = (\langle c_r^\dagger(\vec{p}\,) \rangle)^2\): the occupation operators have only eigenvalues 0 and 1.

To avoid infinite vacuum contributions to the energy and charge the normal product is introduced. The expression for the current density now becomes \(J_\mu = -ieN(\overline{\psi}(\gamma_\mu\psi))\). This product is obtained by:

- Expand all fields into creation and annihilation operators,
- Keep all terms which have no annihilation operators, or in which they are on the right of the creation operators,
- In all other terms interchange the factors so that the annihilation operators go to the right. In an interchange of two fermion operators add a minus sign, in interchange of two boson operators not. Assume thereby that all commutators are zero.

### Quantization of the electromagnetic field

Starting with the Lagrange density \(\mathcal{L} = -\frac{1}{2} \frac{\partial A_\mu}{\partial x_\nu} \frac{\partial A_\nu}{\partial x_\mu}\)

it follows for the field operators \(A(x)\):
\[ A(x) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \frac{1}{\sqrt{2\omega_k}} \sum_{m=1}^4 \left( a_m(\vec{k}) \epsilon^m(\vec{k}) e^{ikx} + a^\dagger(\vec{k}) \epsilon^m(\vec{k})^* e^{-ikx} \right) \]

The operators obey \( [a_m(\vec{k}), a_{m'}^\dagger(\vec{k})] = \delta_{mm'} \delta_{kk'} \). All other commutators are 0. \( \langle m \rangle \) gives the polarization direction of the photon: \( \langle m=1,2 \rangle \) gives transversely polarized, \( \langle m=3 \rangle \) longitudinally polarized and \( \langle m=4 \rangle \) time like polarized photons. Further:

\[ \langle [A_{\mu}(x), A_{\nu}(x')] = i \delta_{\mu\nu} D(x-x') \quad \text{with} \quad D(y) = \Delta(y) |_{m=0} \]

In spite of the fact that \( A_4 = iV \) is imaginary in the classical case, \( A_4 \) is still defined to be Hermitian because otherwise the sign of the energy becomes incorrect. By changing the definition of the inner product in configuration space the expectation values for \( \langle A_{\{1,2,3\}}(x) \rangle \in \mathbb{R} \) and for \( \langle A_4(x) \rangle \) become imaginary.

If the potentials satisfy the Lorentz gauge condition \( \partial_\mu A_\mu = 0 \) the \( (E) \) and \( (B) \) operators derived from these potentials will satisfy the Maxwell equations. However, this gives problems with the commutation rules. It is now demanded that only those states are permitted for which

\[ \langle \partial A^\dagger_{\mu} \rangle |\Phi\rangle = 0 \]

This results in:

\[ \langle \bar{\partial} \rangle \text{left\{ partial A^\dagger_{\mu} \} \text{partial x}_{\mu}\rangle |\Phi\rangle = 0 \]

From this follows that \( \langle (a_3(\vec{k}) a_4(\vec{k})) |\Phi\rangle = 0 \). With a local gauge transform one obtains \( \langle N_3(\vec{k}) \rangle = 0 \) and \( \langle N_4(\vec{k}) \rangle = 0 \). However, this only applies to free EM-fields: for intermediate states in interactions there can exist longitudinal and time like photons. These photons are also responsible for the stationary Coulomb potential.

**Interacting fields and the S-matrix**

The \( \langle S \rangle \) (scattering)-matrix gives a relation between the initial and final states of an interaction: \( \langle \Phi(\infty) | S | \Phi(-\infty) \rangle \). If the Schrödinger equation is integrated:

\[ |\Phi(t)\rangle = |\Phi(-\infty)\rangle e^{-i\int H_{\text{int}}(t_{\text{int}}) dt_{\text{int}}}|\Phi(t_{\text{int}})\rangle \]

and when perturbation theory is applied one finds that:

\[ |S|^4 = \sum_{n=0}^\infty \langle \Phi(\infty) | S_{ij} | \Phi(\infty) \rangle \]

Here, the \( (T) \)-operator means a time-ordered product: the terms in such a product must be ordered in increasing time order from the right to the left so that the earliest terms act first. The \( \langle S \rangle \)-matrix is then given by:

\[ \langle S_{ij} \rangle = \left\langle \Phi_j | \Phi(\infty) \right\rangle \]

The interaction Hamilton density for the interaction between the electromagnetic and the electron-positron field is:

\[ \langle \text{cal} \]
When this is expanded as: \[ {\cal H}_{\text{int}} = i e N (\overline{\psi^+} \gamma_\mu \psi^+ A^-_\mu + \overline{\psi^-} \gamma_\mu \psi^+ A^-_\mu - \overline{\psi^+} A^+_\mu \psi^- - \overline{\psi^-} A^+_\mu \psi^- ) \]
eight terms appear. Each term corresponds to a possible process. The term \( i e \overline{\psi^+} \gamma_\mu \psi^+ A^-_\mu \) acting on \(|\Phi\rangle\) gives transitions where \( A^-_\mu \) creates a photon, \( \psi^+ \) annihilates an electron and \( \overline{\psi^+} \) annihilates a positron. Only terms with the correct number of particles in the initial and final state contribute to a matrix element \( \langle \Phi_i | S | \Phi_j \rangle \). Further the factors in \(|\{\text{cal } H}\rangle_{\text{int}}\rangle\) can create and thereafter annihilate particles: the virtual particles.

The expressions for \( S^{(n)} \) contain time-ordered products of normal products. This can be written as a sum of normal products. The operators that appear describe the minimum changes necessary to change the initial state into the final state. The effects of the virtual particles are described by the (anti)commutator functions. Some time-ordered products are:

\[
\begin{align*}
&\left\{ T \left\{ \Phi(x) \Phi(y) \right\} = N \left\{ \Phi(x) \Phi(y) \right\} + \frac{1}{2} \Delta^{F}(x-y) \\
&\left\{ T \left\{ \psi_\alpha(x) \overline{\psi_\beta(y)} \right\} = N \left\{ \psi_\alpha(x) \overline{\psi_\beta(y)} \right\} - \frac{1}{2} S^{F}_{\alpha \beta}(x-y) \\
&\left\{ T \left\{ A_\mu(x) A_\nu(y) \right\} = N \left\{ A_\mu(x) A_\nu(y) \right\} + \frac{1}{2} \delta_{\mu \nu} D^{F}_{\mu \nu}(x-y) \\
\end{align*}
\]

Here, \( S^{F}(x) = (\gamma_\mu \partial_\mu - M) \Delta^{F}(x) \), \( D^{F}(x) = \Delta^{F}(x)|_{m=0} \) and

\[
\begin{align*}
&\Delta^{F}(x) = \lim_{\epsilon \rightarrow 0} \frac{-2i}{(2 \pi)^4} \int \frac{e^{ikx}}{k^2 + m^2 - i\epsilon} d^4k \\
&\text{and } \quad S^{F}(x) = \lim_{\epsilon \rightarrow 0} \frac{-2i}{(2 \pi)^4} \int e^{ipx} \frac{i\gamma_\mu p_\mu - M}{p^2 + M^2 - i\epsilon} d^4p \\
\end{align*}
\]

The term \( \frac{1}{2} \Delta^{F}(x-y) \) is called the contraction of \( \langle \Phi(x) \rangle \) and \( \langle \Phi(y) \rangle \), and is the expectation value of the time-ordered product in the vacuum state. Wick’s theorem gives an expression for the time-ordered product of an arbitrary number of field operators. The graphical representation of these processes are called Feynman diagrams. In the \( \langle x \rangle \)-representation each diagram describes a number of processes. The contraction functions can also be written as:

\[
\begin{align*}
&\Delta^{F}(x) = \lim_{\epsilon \rightarrow 0} \frac{-2i}{(2 \pi)^4} \int \frac{e^{ikx}}{k^2 + m^2 - i\epsilon} d^4k \\
&\text{and } \quad S^{F}(x) = \lim_{\epsilon \rightarrow 0} \frac{-2i}{(2 \pi)^4} \int e^{ipx} \frac{i\gamma_\mu p_\mu - M}{p^2 + M^2 - i\epsilon} d^4p \\
\end{align*}
\]

In the expressions for \( S^{(2)} \) this gives rise to terms \( \langle \delta(p+k-p'-k') \rangle \). This means that energy and momentum is conserved. However, virtual particles do not obey the relation between energy and momentum.

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**Divergences and renormalization**

It turns out that higher orders contribute infinite terms because only the sum \( p+k \) of the four-momentum of the virtual
particles is fixed. An integration over one of them becomes \(\infty\). In the \(\delta\)-representation this can be understood because the product of two functions containing \(\delta\)-like singularities is not well defined. This is solved by discounting all divergent diagrams in a renormalization of \(e\) and \(M\). It is assumed that an electron, if there, would not create an electromagnetic field, would have a mass \(M_0\) and a charge \(e_0\) unequal to the observed mass \(M\) and charge \(e\).

In the Hamilton and Lagrange density of the free electron-positron field \(\langle M_0 \rangle\) appears. So this gives, with \(\langle M=M_0+\Delta M \rangle\):

\[
\langle \text{cal L}_{\text{e-p}}(x) \rangle = -\overline{\psi(x)}(\gamma_{\mu}\partial_{\mu}+M_0)\psi(x) = -\overline{\psi(x)}(\gamma_{\mu}\partial_{\mu}+M)\psi(x) + \Delta M\overline{\psi(x)}\psi(x) \]

and \(\langle \text{cal H}_{\text{int}} \rangle = ieN(\overline{\psi}\gamma_{\mu}\psi A_{\mu}) - i\Delta eN(\overline{\psi}\gamma_{\mu}\psi A_{\mu})\).

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**Classification of elementary particles**

Elementary particles can be categorized as follows:

1. **Hadrons**: these consist of quarks and can be categorized in:
   1. **Baryons**: these consist of 3 quarks or 3 antiquarks.
   2. **Mesons**: these consist of 1 quark and 1 antiquark.

2. **Leptons**: \(e^\pm, \mu^\pm, \tau^\pm, \nu_e, \nu_\mu, \nu_\tau, \overline{\nu}_e, \overline{\nu}_\mu, \overline{\nu}_\tau\).

3. **Field quanta**: \(\gamma, W^\pm, Z^0, \) gluons, gravitons (?).

4. **Higgs particle**: \(\phi\).

An overview of particles and antiparticles is given in the following table:

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<th>Particle</th>
<th>spin ((\hbar))</th>
<th>B</th>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>e^\pm</td>
<td>1/2</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

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Here $B$ is the baryon number and $L$ the lepton number. It is found that there are three different lepton numbers, one for $e$, $\mu$, and $\tau$, which are separately conserved. $T$ is the isospin, with $T_3$ the projection of the isospin on the third axis, $C$ the charm, $S$ the strange and $B^*$ the bottom quarks. The anti-particles have quantum numbers with the opposite sign except for the total isospin $T$. The composition of the (anti)quarks of the hadrons is given in the following table, together with their mass in MeV in their ground state:

| $(\mu_-)$       | 1/2 | 0 | 1 | 0 | 0 | 0 | 0 |
| $(\tau^-)$      | 1/2 | 0 | 1 | 0 | 0 | 0 | 0 |
| $(\nu_e)$       | 1/2 | 0 | 1 | 0 | 0 | 0 | 0 |
| $(\nu_\mu)$     | 1/2 | 0 | 1 | 0 | 0 | 0 | 0 |
| $(\nu_\tau)$    | 1/2 | 0 | 1 | 0 | 0 | 0 | 0 |
| $(\text{gamma})$| 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| gluon           | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $W^+$           | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| $Z$             | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| graviton        | 2 | 0 | 0 | 0 | 0 | 0 | 0 |
| Higgs           | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

| $(\pi^0)$       | $\frac{1}{2}(\sqrt{2}(u\overline{u}+d\overline{d})$ | 134.9764 | $J/(\Psi)$ | $b\overline{(\text{charm})}$ | 3096.8 | $(\text{anti})$ |
| $(\pi^+)$       | $u\overline{(\text{charm})}$ | 139.56995 | $(\Upsilon)$ | $b\overline{(\text{charm})}$ | 9460.37 | |
| $(\pi^-)$       | $d\overline{(\text{charm})}$ | 139.56995 | $p\overline{(\pi^0)}$ | $u u d$ | 938.27231 | $(\text{anti})$ |
| $K^0$(^0)       | $s\overline{(\text{charm})}$ | 497.672 | $p\overline{(\pi^0)}$ | $\overline{(\text{charm})}$ | 938.27231 | |
Each quark can exist in two spin states. So mesons are bosons with spin 0 or 1 in their ground state, while baryons are fermions with spin $\frac{1}{2}$ or $\frac{3}{2}$. There exist excited states with higher internal $L$. Neutrino’s have a helicity of $\frac{1}{2}$ while antineutrino’s have only $\frac{1}{2}$ as possible values.

The quantum numbers are subject to conservation laws. These can be derived from symmetries in the Lagrange density:
continuous symmetries give rise to additive conservation laws, discrete symmetries result in multiplicative conservation laws.

**Geometrical conservation laws** are invariant under *Lorentz transformations* and the CPT-operation. These are:

1. Mass/energy because the laws of nature are invariant for translations in time.
2. Momentum because the laws of nature are invariant for translations in space.
3. Angular momentum because the laws of nature are invariant for rotations.

**Dynamical conservation laws** are invariant under the CPT-operation. These are:

1. Electrical charge because *Maxwell's equations* are invariant under gauge transforms.
2. Colour charge is conserved.
3. Isospin because QCD is invariant for rotations in T-space.
4. Baryon number and lepton number are conserved but not under a possible SU(5) symmetry of the laws of nature.
5. Quarks type is only conserved under the colour interaction.
6. Parity is conserved except for weak interactions.

The elementary particles can be classified into three families:

<table>
<thead>
<tr>
<th>1st generation</th>
<th>2nd generation</th>
<th>3rd generation</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\text{e}^-)</td>
<td>(\mu^-)</td>
<td>(\tau^-)</td>
</tr>
<tr>
<td>(\text{e}^+)</td>
<td>(\mu^+)</td>
<td>(\tau^+)</td>
</tr>
<tr>
<td>(\nu_e)</td>
<td>(\nu_\mu)</td>
<td>(\nu_\tau)</td>
</tr>
<tr>
<td>(\overline{d})</td>
<td>(\overline{s})</td>
<td>(\overline{b})</td>
</tr>
<tr>
<td>(\overline{u})</td>
<td>(\overline{c})</td>
<td>(\overline{t})</td>
</tr>
</tbody>
</table>

Quarks exist in three colours but because they are *confined* these colours cannot be seen directly. The colour force does *not* decrease with distance. The potential energy will become high enough to create a quark-antiquark pair when it is tried to separate an (anti)quark from a hadron. This will result in two hadrons and not in free quarks.

**P and CP-violation**

It is found that the weak interaction violates P-symmetry, and even CP-symmetry is not conserved. Some processes which violate P symmetry but conserve the combination CP symmetry are:

1. \(\mu^-\)-decay: \(\mu^- \rightarrow e^- + \nu_\mu + \overline{\nu_e}\). Left-handed electrons appear more than \(1000\) times as often as right-handed ones.
2. \((\text{beta})\)-decay of spin-polarized \(^{60}\text{Co}\): \(^{60}\text{Co} \rightarrow ^{60}\text{Ni} + e^- + \nu_e\). More electrons with a spin parallel to the Co than with a spin antiparallel are created: \((\text{parallel}/(\text{antiparallel})/\text{total}) = 20\%\).

3. There is no connection with the neutrino: the decay of the \(\Lambda\) particle through: \(\Lambda \rightarrow p^+ + \pi^-\) and \(\Lambda \rightarrow n^0 + \pi^0\) also has these properties.

The CP-symmetry was found to be violated by the decay of neutral Kaons. These are the lowest possible states with an s-quark so they can decay only weakly. The following holds: \((\text{CP} | \text{K}^0 \rangle = \eta | \overline{\text{K}^0} \rangle\) where \(\eta\) is a phase factor. Further \((\text{CP} | \text{K}^0 \rangle = 2 \text{CP} | \overline{\text{K}^0} \rangle\) because \(\text{CP} | \text{K}^0 \rangle\) and \(\text{CP} | \overline{\text{K}^0} \rangle\) have an intrinsic parity of \((-1)\). From this follows that \(\text{CP} | \text{K}^0 \rangle\) and \(\text{CP} | \overline{\text{K}^0} \rangle\) are not eigenvalues of CP: \(\text{CP} | \text{K}^0 \rangle = \eta | \overline{\text{K}^0} \rangle\). The linear combinations

\[
\frac{1}{2} \sqrt{2} | \text{K}_1^0 \rangle = | \text{K}^0 \rangle + | \overline{\text{K}^0} \rangle
c and
\frac{1}{2} \sqrt{2} | \text{K}_2^0 \rangle = | \text{K}^0 \rangle - | \overline{\text{K}^0} \rangle
d

are eigenstates of CP: \(\text{CP} | \text{K}_1^0 \rangle = | \text{K}_1^0 \rangle\) and \(\text{CP} | \text{K}_2^0 \rangle = | \text{K}_2^0 \rangle\). A basis of \(\text{CP} | \text{K}^0 \rangle\) and \(\text{CP} | \overline{\text{K}^0} \rangle\) is practical while describing weak interactions. For colour interactions a basis of \(\text{CP} | \text{K}^0 \rangle\) and \(\text{CP} | \overline{\text{K}^0} \rangle\) is practical because then the number \(\text{u}(-\text{u})\) of \(\text{u}(-\overline{\text{u}})\) is constant. The expansion postulate must be used for weak decays:

\[
\frac{1}{2} \sqrt{2} | \text{K}_1^0 \rangle = | \text{K}^0 \rangle + | \overline{\text{K}^0} \rangle
\]

The probability of finding a final state with \(\text{CP} = -1\) is \(\left| \langle \text{K}_1^0 | \text{K}^0 \rangle \right|^2\), the probability of \(\text{CP} = +1\) decay is \(\left| \langle \text{K}_2^0 | \text{K}^0 \rangle \right|^2\).

The relation between the mass eigenvalues of the quarks (unaccented) and the fields arising in the weak currents (accented) is \((u',c',t') = (u,c,t)\), and:

\[
\begin{vmatrix}
\cos\theta_1 & \sin\theta_1 & 0 \\
-\sin\theta_1 & \cos\theta_1 & 0 \\
0 & 0 & 1
\end{vmatrix}
\]

The standard model

When one wants to make the Lagrange density which describes a field invariant for local gauge transforms from a certain group, one has to perform the transform

\[
\left(\partial_x \mu \right) = \frac{1}{\hbar} \left( \frac{1}{\hbar} \frac{Dx_\mu}{Dx_\mu} \right) = i \left( \partial_x A_\mu \right)
\]
Here the \( L_k \) are the generators of the gauge group (the “charges”) and the \( A_{\mu}^k \) are the gauge fields. \( g \) is the matching coupling constant. The Lagrange density for a scalar field becomes:

\[
\mathcal{L} = -\frac{1}{2} (D_{\mu}\Phi^*D^{\mu}\Phi + M^2\Phi^*\Phi) - \frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu}
\]

and the field tensors are given by:

\[
F_{\mu\nu}^a \equiv \partial_{\mu} A_{\nu}^a - \partial_{\nu} A_{\mu}^a + g c_{lm} A_{\mu}^l A_{\nu}^m.
\]

### The electroweak theory

The electroweak interaction arises from the necessity to keep the Lagrange density invariant for local gauge transforms of the group SU(2)\( \otimes \)U(1). Right- and left-handed spin states are treated differently because the weak interaction does not conserve parity. If a fifth Dirac matrix is defined by:

\[
\gamma_5 := \gamma_1 \gamma_2 \gamma_3 \gamma_4 = -
\]

the left- and right-handed solutions of the Dirac equation for neutrinos are given by:

\[
\psi_{\text{L}} = \frac{1}{2} (1 + \gamma_5) \psi \quad \text{and} \quad \psi_{\text{R}} = \frac{1}{2} (1 - \gamma_5) \psi
\]

It appears that neutrinos are always left-handed while antineutrinos are always right-handed. The hypercharge \( Y \), for quarks given by \( Y = (Y_{\text{B} + S + C + B^* + T'}) \), is defined by:

\[
Q = \frac{1}{2} Y + T_3
\]

so \([Y, T_k] = 0\).

The group U(1)\( \otimes \)SU(2)\( \otimes \)SU(2) is taken as symmetry group for the electroweak interaction because the generators of this group commute. The multiplets are classified as follows:

<table>
<thead>
<tr>
<th>( T )</th>
<th>( e )</th>
<th>( \nu )</th>
<th>( u )</th>
<th>( d )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>(-1)</td>
<td>(-1)</td>
<td>(-1)</td>
<td>(-1)</td>
<td>(-1)</td>
</tr>
</tbody>
</table>

Now, one field \( B_{\mu}(x) \) is connected with gauge group U(1) and three gauge fields \( \vec{A}_{\mu}(x) \) are connected.
with SU(2). The total Lagrange density (minus the field terms) for the electron-fermion field now becomes:

\[
\begin{aligned}
\mathcal{L}_{0,\text{EW}} &= -\left(\bar{\psi}_{\nu e, L} \gamma^\mu \left(\partial_\mu - i \frac{g}{\hbar} \vec{A}_\mu \cdot \frac{1}{2} \vec{\sigma} \right) - \frac{1}{2} i \frac{g'}{\hbar} B_\mu \right) \left(\begin{array}{c} \psi_{\nu e, L} \\ \psi_{e L} \end{array}\right) - \\
&\bar{\psi}_{e R} \gamma^\mu \left(\partial_\mu - \frac{1}{2} i \frac{g'}{\hbar} (-2) B_\mu \right) \psi_{e R}
\end{aligned}
\]

Here, \( \frac{1}{2} \vec{\sigma} \) are the generators of \( T \) and \(-1\) and \(-2\) the generators of \( Y \).

### Spontaneous symmetry breaking: the Higgs mechanism

All leptons are massless in the equations above. Their mass is probably generated by spontaneous symmetry breaking. This means that the dynamic equations which describe the system have a symmetry which the ground state does not have. It is assumed that there exists an isospin-doublet of scalar fields \( \Phi \) with electrical charges +1 and 0 and potential \( V(\Phi) = -\mu^2 \Phi^* \Phi + \lambda (\Phi^* \Phi)^2 \). Their antiparticles have charges \(-1\) and 0. The extra terms in \( \mathcal{L} \) arising from these fields, \( \mathcal{L}_H = (D_{L \mu} \Phi)^* (D_L^\mu \Phi) - V(\Phi) \), are globally U(1)\( \otimes \)SU(2) symmetric. Hence the state with the lowest energy corresponds with the state \( (\Phi^* (x) \Phi(x) = v^2/2 \Phi(0) \Phi(0)) \), and \( |0\rangle \rightarrow |\Phi \rangle \). The field can be written (where \( \omega^\pm \) and \( z \) are Nambu-Goldstone bosons which can be transformed away, and \( m_\phi = \mu \sqrt{2} \)) as:

\[
\Phi = \left(\begin{array}{c} \Phi^+ \\ \Phi^0 \end{array}\right) = \left(\begin{array}{c} i \omega^+ \\ (v + \phi - iz)/\sqrt{2} \end{array}\right)
\]

Because this expectation value \( \neq 0 \) the SU(2) symmetry is broken but the U(1) symmetry is not. When the gauge fields in the resulting Lagrange density are separated one obtains:

\[
\begin{aligned}
W_\mu^- &= \frac{1}{2} \sqrt{2}(A_\mu^1 + i A_\mu^2) \\
W_\mu^+ &= \frac{1}{2} \sqrt{2}(A_\mu^1 - i A_\mu^2) \\
Z_\mu &= \frac{g A_\mu^3 - g' B_\mu}{\sqrt{g^2 + g'^2}} = A_\mu^3 \cos(\theta_W) - B_\mu \sin(\theta_W) \\
A_\mu &= \frac{g' A_\mu^3 + g B_\mu}{\sqrt{g^2 + g'^2}} = A_\mu^3 \sin(\theta_W) + B_\mu \cos(\theta_W)
\end{aligned}
\]

where \( (\theta_W) \) is called the Weinberg angle. For this angle: \( \sin^2(\theta_W) = 0.255 \pm 0.010 \). Relations for the masses of the field quanta can be obtained from the remaining terms: \( M_W = \frac{1}{2} gg' \) and \( M_Z = \frac{1}{2} \sqrt{g^2 + g'^2} \), and for the elementary charge: \( e = \frac{g}{\sqrt{g^2 + g'^2}} \). Experimentally it is found that \( M_W = 80.022 \pm 0.010 \text{ GeV/c}^2 \) and \( M_Z = 91.187 \pm 0.007 \text{ GeV/c}^2 \). According to the weak theory this should be: \( M_W = 83.0 \pm 0.24 \text{ GeV/c}^2 \) and \( M_Z = 93.8 \pm 2.0 \text{ GeV/c}^2 \).
Quantum chromodynamics

Coloured particles interact because the Lagrange density is invariant for the transformations of the group SU(3) of the colour interaction. A distinction can be made between two types of particles:

1. “White” particles: they have no colour charge, the generator \( \langle \vec{T} \rangle = 0 \).
2. “Coloured” particles: the generators \( \langle \vec{T} \rangle \) are eight \( \langle 3 \times 3 \rangle \) matrices. There exist three colours and three anticolours.

The Lagrange density for coloured particles is given by

\[
\mathcal{L}_{\text{QCD}} = i \sum_k \overline{\Psi_k} \gamma^\mu D_\mu \Psi_k + \sum_{k,l} \overline{\Psi_k} M_{kl} \Psi_l - \frac{1}{4} F^a_{\mu\nu} F_a^{\mu\nu}
\]

The gluons remain massless because this Lagrange density does not contain spinless particles. Because left- and right-handed quarks now belong to the same multiplet a mass term can be introduced. This term can be brought in the form \( \langle M_{kl} \rangle = m_k \delta_{kl} \).

Path integrals

Besides the Schrödinger equation the development in time of a quantum mechanical system can also be described by a path integral (Feynman):

\[
\langle \psi(x',t') | \psi(x,t) \rangle = \int F(x',t',x,t) \psi(x,t) dx
\]

in which \( \langle F(x',t',x,t) \rangle \) is the amplitude of probability to find a system at time \( \langle t' \rangle \) and \( \langle x' \rangle \) if it was at \( \langle x \rangle \) at time \( \langle t \rangle \). Then,

\[
\langle F(x',t',x,t) \rangle = \int \exp \left( \frac{iS[x]}{\hbar} \right) dx
\]

where \( \langle S[x] \rangle \) is an action-integral: \( \langle S[x] \rangle = \int L(x,\dot{x},t) dt \). The notation \( \langle d[x] \rangle \) means that the integral has to be taken over all possible paths \( \langle [x] \rangle \):

\[
\langle d[x] \rangle = \lim_{\n \rightarrow \infty} \prod_n \left\{ \int_{-\infty}^{\infty} dx(t_n) \right\}
\]

in which \( \langle N \rangle \) is a normalization constant. A probability amplitude \( \langle \exp(iS/\hbar) \rangle \) is assigned to each path. The classical limit can be found by taking \( \langle \delta S = 0 \rangle \): the average of the exponent vanishes, except where it is stationary. In quantum field theory, the probability of the transition of a field operator \( \langle \Phi(x, \vec{r}, t) \rangle \) to \( \langle \Phi'(x, \vec{r}, t) \rangle \) is given by

\[
\langle F(\Phi'(x, \vec{r}, t), \Phi(x, \vec{r})) \rangle = \int \exp \left( \frac{iS[\Phi]}{\hbar} \right) \Phi(\vec{r}) d^4x
\]

with the action-integral \( \langle S[\Phi] \rangle = \int d^4x \Omega \mathcal{L} \). 

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Unification and quantum gravity

The strength of forces varies with energy and the reciprocal coupling constants approach each other with increasing energy. The SU(5) model predicts complete unification of the electromagnetic, weak and colour forces at \(10^{15}\) GeV. It also predicts twelve extra X bosons which couple leptons and quarks and are e.g. responsible for proton decay, with the dominant channel \(\{\text{p}^+\rightarrow\pi^0+\text{e}^+\}\), and an average lifetime of the proton of \(10^{31}\) year. This model has been experimentally falsified.

Supersymmetric models assume a symmetry between bosons and fermions and predict partners for the currently known particles with a spin which differs by \(\frac{1}{2}\). The supersymmetric SU(5) model predicts unification at \(10^{16}\) GeV and an average lifetime of the proton of \(10^{33}\) year. The dominant decay channels in this theory are \(\{\text{p}^+\rightarrow\text{K}^++\overline{\text{nu}}_\mu\}\) and \(\{\text{p}^+\rightarrow\text{K}^0+\mu^+\}\).

Quantum gravity only plays a role in particle interactions at the Planck dimensions, where \(\lambda_C\approx R_S\): \(m_{\text{Pl}}=\sqrt{hc/G}=3\cdot10^{19}\) GeV, \(t_{\text{Pl}}=h/m_{\text{Pl}}c^2=\sqrt{hG/c^5}=10^{-43}\) sec and \(r_{\text{Pl}}=ct_{\text{Pl}}\approx 10^{-35}\) m.