13.2: Adiabatic Invariance and Quantum Mechanics

This finding, the invariance of \( \frac{E}{\omega} \) for slow variation of the potential strength in a simple harmonic oscillator, connects directly with quantum mechanics, as was first pointed out by Einstein in 1911. Suppose the (quantum mechanical) oscillator is in the energy eigenstate with \( E = (n + \frac{1}{2}) \hbar \omega \). Then the spatial wave function has \( n \) zeros. If the potential is changed slowly enough (meaning little change over one cycle of oscillation) the oscillator will not jump to another eigenstate (or, more precisely, the probability will go to zero with the speed of change). The wave function will gradually stretch (or compress) but the number of zeroes will not change. Therefore the energy will stay at \( (n + \frac{1}{2}) \hbar \omega \), and track with \( \omega \). Of course, the classical system is a little different: the quantum system is “locked in” to a particular state if the perturbation has vanishingly small frequency components corresponding to the energy differences \( \hbar \omega \) to available states. The classical system, on the other hand, can move to states arbitrarily close in energy. Landau gives a nontrivial analysis of the classical system, concluding that the change in the adiabatic “invariant” is of order \( e^{-\omega_0 \tau} \) for an external change acting over a time \( \tau \).