2.5: Fastest Curve for Given Horizontal Distance

Suppose we want to find the curve a bead slides down to minimize the time from the origin to some specified horizontal displacement $X$, but we don’t care what vertical drop that entails.

Recall how we derived the equation for the curve:

At the minimum, under any infinitesimal variation $\delta y(x)$.

\[
\delta J[y] = \int_{x_{1}}^{x_{2}} \left[ \frac{\partial f(y, y')}{\partial y} \delta y(x) + \frac{\partial f(y, y')}{\partial y'} \delta y'(x) \right] dx = 0
\]

Writing $\delta y' = \delta (dy/dx) = (d/dx) \delta y$, and integrating the second term by parts,

\[
\delta J[y] = \int_{x_{1}}^{x_{2}} \left[ \frac{\partial f(y, y')}{\partial y} - \frac{d}{dx} \left( \frac{\partial f(y, y')}{\partial y'} \right) \right] \delta y(x) dx + \left[ \frac{\partial f(y, y')}{\partial y'} \delta y(x) \right]_{0}^{X} = 0
\]

In the earlier treatment, both endpoints were fixed, $\delta y(0) = \delta y(X) = 0$ so we dropped that final term.

However, we are now trying to find the fastest time for a given horizontal distance, so the final vertical distance is an adjustable parameter: $\delta y(X) \neq 0$

As before, since $\delta J[y]=0$ or arbitrary $\delta y$, we can still choose a $\delta y(x)$ which is only nonzero near some point not at the end, so we must still have

\[
\int \left[ \frac{\partial f(y, y')}{\partial y} \right] \left[ \frac{\partial f(y, y')}{\partial y'} \right] (d/dx) \delta y(x) dx = 0
\]
However, we must also have \( \frac{\partial f(y(X), y'(X))}{\partial y'} \delta y(X) = 0 \), to first order for arbitrary infinitesimal \( \delta y(X) \), (imagine a variation \( \delta y \) only nonzero near the endpoint), this can only be true if \( \frac{\partial f(y, y')}{\partial y'} = 0 \) at \( x=X \)

For the brachistochrone,

\[
[f = \sqrt{\frac{1+y'^2}{2gy}}, \quad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{2gy\left(1+y'^2\right)}}]
\]

so \( \frac{\partial f(y, y')}{\partial y'} = 0 \) at \( x=X \text{ means that } f' = 0 \), the curve is horizontal at the end \( x=X \)

So the curve that delivers the bead a given horizontal distance the fastest is the half-cycloid (inverted) flat at the end. It’s easy to see this fixes the curve uniquely: think of the curve as generated by a rolling wheel, one half-turn of the wheel takes the top point to the bottom in distance \( x=X \)

*Exercise:* how low does it go?