23.2: Driven Damped Pendulum- The Road to Chaos

After the anharmonic oscillator treated in the previous lecture, our second example of a one-dimensional nonlinear system is a driven damped pendulum: that just means replacing the potential term \(-\omega_0^2 x^2\) in the linear oscillator with \(-\omega_0^2 \sin x\), or rather \(-\omega_0^2 \sin \phi\), to make clear we have an angular system.

For small amplitude oscillations, the free pendulum’s frequency is almost independent of amplitude, as has been known for centuries, and the first correction on increasing the amplitude is from a quartic addition to the potential, just as in the nonlinear case discussed earlier, except now with a definite value, and an opposite sign, so the (undamped, undriven) frequency decreases with amplitude.

Provided the amplitude remains small, the driven damped pendulum behaves like the driven damped linear oscillator we discussed earlier. But things get much more interesting as the amplitude increases.

For the driven pendulum, the natural measure of the driving force is its ratio to the weight \((m g)\) Taylor calls this the drive strength, so for driving force

\[ F(t) = F_0 \cos \omega t \]

the drive strength

\[ \gamma = \frac{F_0}{mg}, \quad \omega_0 = \sqrt{\frac{g}{L}} \]

The equation of motion (with resistive damping force \(-bv\) and hence resistive torque \(-bL^2 \phi\))

\[ mL^2 \ddot{\phi} = -bL^2 \dot{\phi} - mgL \sin \phi + LF(t) \]
Dividing by $(m L^2)$ and writing the damping term $b/m = 2\beta$ (to coincide with Taylor’s notation, his equation 12.12) we get

$$\ddot{\phi} + 2\beta \dot{\phi} + \omega_0^2 \sin \phi = \gamma \omega_0^2 \cos \omega_0 t$$

(Compare with our equation in the previous lecture for the anharmonic oscillator in Landau notation: $\ddot{x} + 2\lambda \dot{x} + \omega_0^2 x = (f/m) \cos \omega t - \beta x^3$.)