9.2: Index of Refraction

Matter is composed of electric charges. This is something of a miracle. We cannot understand it without quantum mechanics. In a purely classical world, there would be no stable atoms or molecules. Because of quantum mechanics, the world does not collapse and we can build stable chunks of matter composed of equal numbers of positive and negative charges. In a chunk of matter in equilibrium, the charge and current are very close to zero when averaged over any large smooth region. However, in the presence of external electric and magnetic fields, such as those produced by an electromagnetic wave, the charges out of which the matter is built can move. This gives rise to what are called “bound” charges and currents, distinguishable from the “free” charges that are not part of the matter itself. These bound charges and currents affect the relation between electric and magnetic fields. In a homogeneous and isotropic material, which is a fancy way of describing a material that does not have any preferred axis, the effects of the matter (averaged over large regions) can be incorporated by replacing the constants $\epsilon_0$ and $\mu_0$ by the permittivity and permeability, $\epsilon$ and $\mu$. Then Maxwell’s equations for electromagnetic waves, (8.35)-(8.37), are modified to

\[
\begin{align*}
\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= -\frac{\partial B_z}{\partial t}, \\
\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= -\frac{\partial B_x}{\partial t}, \\
\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= -\frac{\partial B_y}{\partial t}, \\
\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} &= \mu \epsilon \frac{\partial E_z}{\partial t}, \\
\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} &= \mu \epsilon \frac{\partial E_x}{\partial t}, \\
\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} &= \mu \epsilon \frac{\partial E_y}{\partial t}, \\
\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} &= 0,
\end{align*}
\]
Now (8.41)-(8.47) are satisfied with the appropriate substitutions, $\epsilon_0 \rightarrow \epsilon, \quad \mu_0 \rightarrow \mu$.

In particular, the dispersion relation, (8.47), becomes $\omega^2 = \frac{1}{\mu \epsilon} k^2 = \frac{\mu_0 \epsilon_0}{\mu \epsilon} c^2 k^2$.

so electromagnetic waves propagate with velocity $v = \frac{\omega}{k} = c \sqrt{\frac{\mu_0 \epsilon_0}{\mu \epsilon}}$.

and (8.48) becomes $\beta_y^{\pm} = \pm \sqrt{\mu \epsilon} \varepsilon_x^{\pm}, \quad \beta_x^{\pm} = \mp \sqrt{\mu \epsilon} \varepsilon_y^{\pm}$.

The factor $n = \sqrt{\frac{\mu \epsilon}{\mu_0 \epsilon_0}}$ is called the index of refraction of the material. $1/n$ is the ratio of the speed of light in the material to the speed of light in vacuum. In terms of $1/n$, we can write (9.52) as $\beta_y^{\pm} = \pm \frac{n}{c} \varepsilon_x^{\pm}, \quad \beta_x^{\pm} = \mp \frac{n}{c} \varepsilon_y^{\pm}$.

Note also that we can rewrite (9.50) in the following useful form: $k = n \frac{\omega}{c}$.

For fixed frequency, the wave number is proportional to the index of refraction. For most transparent materials, $1/n$ is very close to 1, and can be ignored. But $\epsilon$ can be very different from 1, and is often quite important. For example, the index of refraction of ordinary glass is about 1.5 (it varies slightly with frequency, but we will discuss the interesting and familiar consequences of this later, when we treat waves in three dimensions).

**Reflection from a Dielectric Boundary**

Let us now consider a plane wave in the $(+z)$ direction in a universe that is filled with a dielectric material with index of refraction $n = \sqrt{\epsilon / \epsilon_0}$, for $z < 0$ and filled with another dielectric material with index of refraction $n' = \sqrt{\epsilon' / \epsilon_0}$, for $z > 0$. The boundary between the two dielectrics, the plane $z = 0$, is analogous to the boundary between two regions of the rope in Figure 9.1. We would, therefore, expect some reflection from this surface.

Because the electric field in a plane electromagnetic wave is perpendicular to its direction of motion, we know that in this case that it is in the $(x,y)$ plane. It doesn’t matter in what direction the electric field of our incoming plane wave is pointing in the $(x,y)$ plane. That is clear by symmetry. The system looks the same if we rotate it around the $(z)$ axis, thus we can always rotate until our $(\vec{e}_+) = \pm \varepsilon (x)$ vector is pointing in some convenient direction, say the $(x)$ direction. It is then pretty obvious that the reflected and transmitted waves will also have their electric fields in the $(\pm x)$ direction. Actually, we can turn this into a symmetry argument too. If we reflect the system in the $(x,y)$ plane, both the incoming wave and the dielectric are unchanged, but any $(y)$ component of the transmitted or reflected waves would change sign. Thus these

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components must vanish, by symmetry. Magnetic fields work the other way, because of the cross product of vectors in their definition. Thus we can write

\[
\begin{aligned}
E_x(z, t) &= A e^{i(k z - \omega t)} + R A e^{-i(k z - \omega t)} \quad \text{for } z < 0, \\
B_y(z, t) &= \frac{n}{c} A e^{i(k z - \omega t)} - \frac{n}{c} R A e^{-i(k z - \omega t)}
\end{aligned}
\]

and

\[
\begin{aligned}
E_x(z, t) &= \tau A e^{i(k z - \omega t)} \\
B_y(z, t) &= \frac{n'}{c} \tau A e^{i(k z - \omega t)}
\end{aligned} \quad \text{for } z > 0,
\]

where we have continued our convention of calling the amplitude of the incoming wave \(A\). Here, \(\langle A \rangle\) has units of electric field. In (9.56) and (9.57), we have used (9.54) to get the \(B\) field from the \(E\) field.

To compute \(\langle R \rangle\) and \(\langle \tau \rangle\), we need the boundary conditions at \(z = 0\). For this we go back to Maxwell. The only way to get a discontinuity in the electric field is to have a sheet of charge. In a dielectric, charge builds up on the boundary only if there is a polarization perpendicular to the boundary. In this case, the electric fields, and therefore the polarizations, are parallel to the boundary, and thus the \(\langle E \rangle\) field is continuous at \(z = 0\). The only way to get a discontinuity of the magnetic field, \(\langle B \rangle\), is to have a sheet of current. If \(\langle \mu \rangle\) were not equal to 1 in one of the materials, then we would have a nonzero magnetization, and we would have to worry about current sheets at the boundary. However, because these are only dielectrics, and \(\langle \mu = 1 \rangle\) in both, there is no magnetization and the \(\langle B \rangle\) field is continuous at \(z = 0\) as well. Thus we can immediately read off the boundary conditions: \(1 + R = \tau, \quad n(1 - R) = n' \tau\).

Because of (9.55), the boundary condition (9.58) is equivalent to

\[
\begin{aligned}
1 + R &= \tau, \\
\quad k(1 - R) &= k' \tau
\end{aligned}
\]

which looks exactly like (9.9) and (9.10). We can simply take over the results of (9.11), \(\tau = \frac{2}{1 + k' / k}, \quad R = \frac{1 - k' / k}{1 + k' / k}\).

\[1\text{See Purcell, chapter 10.}\]