9.5: Conservation of Linear Momentum (Part 1)

Learning Objectives

- Explain the meaning of “conservation of momentum”
- Correctly identify if a system is, or is not, closed
- Define a system whose momentum is conserved
- Mathematically express conservation of momentum for a given system
- Calculate an unknown quantity using conservation of momentum

Recall Newton’s third law: When two objects of masses m_1 and m_2 interact (meaning that they apply forces on each other), the force that object 2 applies to object 1 is equal in magnitude and opposite in direction to the force that object 1 applies on object 2. Let:

- $\vec{F}_{21}$ = the force on m_1 from m_2
- $\vec{F}_{12}$ = the force on m_2 from m_1

Then, in symbols, Newton’s third law says

$$\begin{split} \vec{F}_{21} & = - \vec{F}_{12} \\ m_{1} \vec{a}_{1} & = -m_{2} \vec{a}_{2} \ldotp \end{split} \label{9.10}$$

(Recall that these two forces do not cancel because they are applied to different objects. F_{21} causes m_1 to accelerate, and F_{12} causes m_2 to accelerate.)

Although the magnitudes of the forces on the objects are the same, the accelerations are not, simply because the masses (in

general) are different. Therefore, the changes in velocity of each object are different:

$$\frac{d \vec{v}_{1}}{dt} \neq \frac{d \vec{v}_{2}}{dt} \ldotp$$

However, the products of the mass and the change of velocity are equal (in magnitude):

$$m_{1} \frac{d \vec{v}_{1}}{dt} = - m_{2} \frac{d \vec{v}_{2}}{dt} \ldotp \label{9.11}$$

It's a good idea, at this point, to make sure you're clear on the physical meaning of the derivatives in Equation 9.3.3. Because of the interaction, each object ends up getting its velocity changed, by an amount dv. Furthermore, the interaction occurs over a time interval dt, which means that the change of velocities also occurs over dt. This time interval is the same for each object.

Let's assume, for the moment, that the masses of the objects do not change during the interaction. (We'll relax this restriction later.) In that case, we can pull the masses inside the derivatives:

$$\frac{d}{dt} (m_{1} \vec{v}_{1}) = - \frac{d}{dt} (m_{2} \vec{v}_{2}) \label{9.12}$$

and thus

$$\frac{d \vec{p}_{1}}{dt} = - \frac{d \vec{p}_{2}}{dt} \ldotp \label{9.13}$$

This says that **the rate at which momentum changes is the same for both objects**. The masses are different, and the changes of velocity are different, but the rate of change of the product of m and $\vec{v}$ are the same.

Physically, this means that during the interaction of the two objects (m_1 and m_2), both objects have their momentum changed; but those changes are identical in magnitude, though opposite in sign. For example, the momentum of object 1 might increase, which means that the momentum of object 2 decreases by exactly the same amount.

In light of this, let's re-write Equation \ref{9.12} in a more suggestive form:

$$\frac{d \vec{p}_{1}}{dt} + \frac{d \vec{p}_{2}}{dt} = 0 \ldotp \label{9.14}$$

This says that during the interaction, although object 1's momentum changes, and object 2's momentum also changes, these two changes cancel each other out, so that the total change of momentum of the two objects together is zero.

Since the total combined momentum of the two objects together never changes, then we could write

$$\frac{d}{dt} (\vec{p}_{1} + \vec{p}_{2}) = 0 \label{9.15}$$

from which it follows that

$$\vec{p}_{1} + \vec{p}_{2} = \text{constant} \ldotp \label{9.16}$$

As shown in Figure $\PageIndex{1}$, the total momentum of the system before and after the collision remains the same.

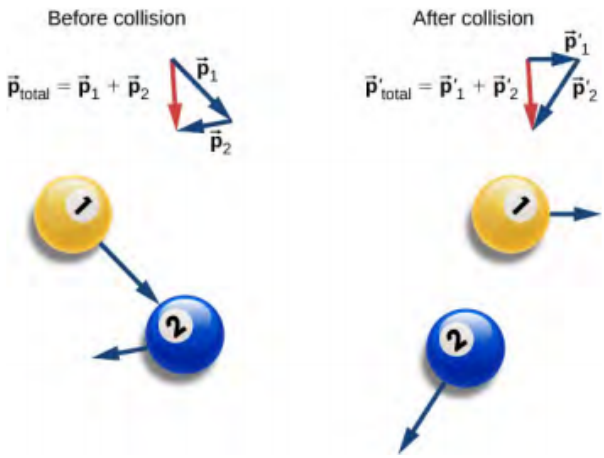

Figure \(\PageIndex{1}\): Before the collision, the two billiard balls travel with momenta \(\vec{p}_{1}\) and \(\vec{p}_{2}\). The total momentum of the system is the sum of these, as shown by the red vector labeled \(\vec{p}_{total}\) on the left. After the collision, the two billiard balls travel with different momenta \(\vec{p}'_{1}\) and \(\vec{p}'_{2}\). The total momentum, however, has not changed, as shown by the red vector arrow \(\vec{p}'_{total}\) on the right.

Generalizing this result to N objects, we obtain

\[\begin{align} \vec{p}_{1} + \vec{p}_{2} + \vec{p}_{3} + \cdots + \vec{p}_{N} &= constant \\ \sum_{j = 1}^{N} \vec{p}_{j} &= constant \ldotp \label{9.17} \end{align} \]

Equation \ref{9.17} is the definition of the total (or net) momentum of a system of N interacting objects, along with the statement that the total momentum of a system of objects is constant in time—or better, is conserved.

Conservation Laws

If the value of a physical quantity is constant in time, we say that the quantity is conserved.

Requirements for Momentum Conservation

There is a complication, however. A system must meet two requirements for its momentum to be conserved:

1. **The mass of the system must remain constant during the interaction.** As the objects interact (apply forces on each other), they may transfer mass from one to another; but any mass one object gains is balanced by the loss of that mass from another. The total mass of the system of objects, therefore, remains unchanged as time passes: \[\Big[\frac{dm}{dt} \Big]_{system} = 0 \ldotp$$
2. **The net external force on the system must be zero**. As the objects collide, or explode, and move around, they exert forces on each other. However, all of these forces are internal to the system, and thus each of these internal forces is balanced by another internal force that is equal in magnitude and opposite in sign. As a result, the change in momentum caused by each internal force is cancelled by another momentum change that is equal in magnitude and opposite in direction. Therefore, internal forces cannot change the total momentum of a system because the changes sum to zero. However, if there is some external force that acts on all of the objects (gravity, for example, or friction), then this force changes the momentum of the system as a whole; that is to say, the momentum of the system is changed by the external force. Thus, for the momentum of the system to be conserved, we must have $$\vec{F}_{ext} = \vec{0} \ldotp$$

A system of objects that meets these two requirements is said to be a **closed system** (also called an isolated system). Thus, the more compact way to express this is shown below.

Law of Conservation of Momentum

The total momentum of a closed system is conserved:

$$\sum_{j = 1}^{N} \vec{p}_{j} = \text{constant} \ldotp$$

This statement is called the **Law of Conservation of Momentum**. Along with the conservation of energy, it is one of the foundations upon which all of physics stands. All our experimental evidence supports this statement: from the motions of galactic clusters to the quarks that make up the proton and the neutron, and at every scale in between. In **a closed system, the total momentum never changes**.

Note that there absolutely **can** be external forces acting on the system; but for the system's momentum to remain constant, these external forces have to cancel, so that the **net** external force is zero. Billiard balls on a table all have a weight force acting on them, but the weights are balanced (canceled) by the normal forces, so there is no net force.

The Meaning of 'System'

A **system** (mechanical) is the collection of objects in whose motion (kinematics and dynamics) you are interested. If you are analyzing the bounce of a ball on the ground, you are probably only interested in the motion of the ball, and not of Earth; thus, the ball is your system. If you are analyzing a car crash, the two cars together compose your system (Figure \(\PageIndex{2}\)).

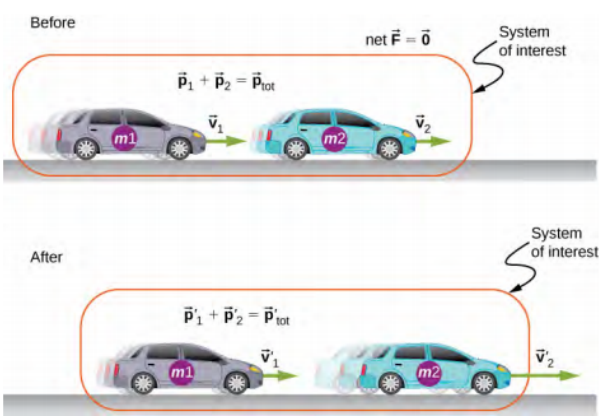

Figure \(\PageIndex{2}\): The two cars together form the system that is to be analyzed. It is important to remember that the contents (the mass) of the system do not change before, during, or after the objects in the system interact.