15.10: Gauge Transformations

We recall (equation 9.1.1) that a static electric field $\mathbf{E}$ can be derived from the negative of the gradient of a scalar potential function of space:

$$\mathbf{E} = -\textbf{grad} \, V.$$ \hfill (15.10.1) \hfill \label{15.10.1}

The zero of the potential is arbitrary. We can add any constant (with the dimensions of potential) to $V$. For example, if we define $V' = V + C$ where $C$ is a constant (in the sense that it is not a function of $x, y, z$) then we can still calculate the electric field from $\mathbf{E} = -\textbf{grad} \, V'$. 

We also recall (Equation 9.2.1) that a static magnetic field $\mathbf{B}$ can be derived from the curl of a magnetic vector potential function:

$$\mathbf{B} = \textbf{curl} \, \mathbf{A}.$$ \hfill (15.10.2) \hfill \label{15.10.2}

Let us also recall here the concept of the $B$-flux from Equation 6.10.1:

$$\Phi_B = \iint \mathbf{B} \cdot d\mathbf{A}.$$ \hfill (15.10.3) \hfill \label{15.10.3}

It will be worth while here to recapitulate the dimensions and SI units of these quantities:

<table>
<thead>
<tr>
<th>Observable</th>
<th>Dimensions</th>
<th>Si Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>$\text{MLT}^{-2}\text{Q}^{-1}$</td>
<td>$\text{V m}^{-1}$</td>
</tr>
<tr>
<td>$B$</td>
<td>$\text{MT}^{-1}\text{Q}^{-1}$</td>
<td>$\text{T}$</td>
</tr>
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</table>
Equation \(\ref{15.10.2}\) is also true for a nonstatic field. Thus a time-varying magnetic field can be represented by the \(\text{curl}\) of a time-varying magnetic vector potential. However, we know from the phenomenon of electromagnetic induction that a varying magnetic field has the same effect as an electric field, so that, if the fields are not static, the electric field is the result of an electrical potential gradient and a varying magnetic field, so that equation \(\ref{15.10.1}\) holds only for static fields.

If we combine the Maxwell equation \(\text{curl} \ A = -\dot{bf[B]}\) with the equation for the definition of the magnetic vector potential \(\text{curl} \ A = bf[B]\), we obtain \(\text{curl} \ (bf[E] + \dot{bf[A]}) = 0\). Then, since \(\text{curl} (bf[\text{grad}])\) of any scalar function is zero, we can define a potential function \(V\) such that

\[
\text{E} + \dot{bf[A]} = -\text{grad} \ V \tag{15.10.4} \label{15.10.4}
\]

(We could have chosen a plus sign, but we choose a minus sign so that it reduces to the familiar \(\text{E} = -\text{grad} \ V\) for a static field.) Thus equations \(\ref{15.10.4}\) and \(\ref{15.10.2}\) define the electric and magnetic potentials – or at least they define the \(\text{grad}\) of \(V\) and the \(\text{curl}\) of \(bf[A]\). But we recall that, in the static case, we can add an arbitrary constant to \(V\) (as long as the constant is dimensionally similar to \(V\)), and the equation \(\text{E} = -\text{grad} \ V\), \(\text{curl} \ A = bf[B]\), still holds. Can we find a suitable transformation for \(V\) and \(bf[A]\) such that equations \(\ref{15.10.2}\) and \(\ref{15.10.4}\) still hold in the nonstatic case? Such a transformation would be a gauge transformation.

Let \(\chi\) be some arbitrary scalar function of space and time. I demand little of the form of \(\chi\); indeed I demand only two things. One is that it is a “well-behaved” function, in the sense that it is everywhere and at all times single-valued, continuous and differentiable. The other is that it should have dimensions \(\text{ML}^2\text{T}^{-1}\text{Q}^{-1}\). This is the same as the dimensions of magnetic flux, but I am not sure that it is particularly helpful to think of this. It will, however, be useful to note that the dimensions of \(\text{grad} \ \chi\) and of \(\dot{\chi}\) are, respectively, the same as the dimensions of magnetic vector potential \(bf[A]\) and of electric potential \(V\).

Let us make the transformations

\[
\text{A}^\prime = \text{A} - \text{grad} \ \chi \tag{15.10.5} \label{15.10.5}
\]

and

\[
V^\prime = V + \dot{\chi} \tag{15.10.6} \label{15.10.6}
\]

We shall see very quickly that this transformation (and we have a wide choice in the form of \(c\)) preserves the forms of
equations \(\text{ref{15.10.2}}\) and \(\text{ref{15.10.4}}\), and therefore this transformation (or, rather, these transformations, since \(c\) can have any well-behaved form) are gauge transformations.

Thus \(\text{curl} \, \textbf{A} = \textbf{B}\) becomes \(\text{curl} \, ( \textbf{A} + \textbf{grad} \chi) = \textbf{B}\). And since \(\text{curl} \, \text{grad}\) of any scalar field is zero, this becomes \(\text{curl} \, \textbf{A} = \textbf{B}\).

Also, \(\text{grad} V = - (\textbf{E} + \dot{\textbf{A}})\) becomes \(\text{grad}(V - \dot{\chi}) = - (\textbf{E} + \dot{\textbf{A}} + \textbf{grad} \dot{\chi})\) or \(\text{grad} V = - (\textbf{E} + \dot{\textbf{A}})\).

Thus the form of the equations is preserved. If we make a gauge transformation to the potentials such as equations \(\text{ref{15.10.5}}\) and \(\text{ref{15.10.6}}\), this does not change the fields \(\textbf{E}\) and \(\textbf{B}\), so that the fields \(\textbf{E}\) and \(\textbf{B}\) are gauge invariant. Maxwell’s equations in their usual form are expressed in terms of \(\textbf{E}\) and \(\textbf{B}\), and are hence gauge invariant.

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