5.19: Charging a Capacitor Through a Resistor

This time, the charge on the capacitor is increasing, so the current, as drawn, is \((\dot{Q})\).

Thus

\[
V - \dot{Q}R - \frac{Q}{C} = 0 \quad \text{(5.19.1)}
\]

Whence:

\[
\int_0^Q \frac{dQ}{CV-Q} = \frac{1}{RC} \int_0^t dt \quad \text{(5.19.2)}
\]

Remember that, at any finite \(t\), \(Q\) is less than its asymptotic value \((CV)\), and you want to keep the denominator of the
Upon integrating Equation \ref{5.19.2}, we obtain
\[Q = CV \left( 1 - e^{-t/(RC)} \right). \tag{5.19.3} \]

Thus the charge on the capacitor asymptotically approaches its final value \(CV\), reaching 63\% \((1 - e^{-1})\) of the final value in time \((RC)\) and half of the final value in time \((RC \ln 2 = 0.6931\) RC\).

The potential difference across the plates increases at the same rate. Potential difference cannot change instantaneously in any circuit containing capacitance.

How does the current change with time? This is found by differentiating Equation \ref{5.19.3} with respect to time, to give
\[I = \frac{V}{R} e^{-t/(RC)}.\]

This suggests that the current grows instantaneously from zero to \((V/R)\) as soon as the switch is closed, and then it decays exponentially, with time constant \((RC)\), to zero. Is this really possible? It is possible in principle if the inductance (see Chapter 12) of the circuit is zero. But the inductance of any closed circuit cannot be exactly zero, and the circuit, as drawn without any inductance whatever, is not achievable in any real circuit, and so, in a real circuit, there will not be an instantaneous change of current. Section 10.15 will deal with the growth of current in a circuit that contains both capacitance and inductance as well as resistance.

### Energy considerations

When the capacitor is fully charged, the current has dropped to zero, the potential difference across its plates is \((V)\) (the EMF of the battery), and the energy stored in the capacitor (see Section 5.10) is
\[
\frac{1}{2}CV^2 = \frac{1}{2}QV.
\]

But the energy lost by the battery is \((QV)\). Let us hope that the remaining \((\frac{1}{2}QV)\) is heat generated in and dissipated by the resistor. The rate at which heat is generated by current in a resistor (see Chapter 4 Section 4.6) is \((I^2R)\). In this case, according to the previous paragraph, the current at time \(t\) is
\[
I = \frac{V}{R} e^{-t/(RC)}.
\]

so the total heat generated in the resistor is
\[
\int_0^\infty e^{-2t/(RC)} dt = \frac{1}{2}CV^2,
\]

so all is well. The energy lost by the battery is shared equally between \((R)\) and \((C)\).

**Neon lamp**
Here’s a way of making a neon lamp flash periodically.

In Figure \((V.\frac{1}{2})\) (sorry about the fraction – I slipped the Figure in as an afterthought!), the thing that looks something like a happy face on the right is a discharge tube; the dot inside it indicates that it’s not a complete vacuum inside, but it has a little bit of gas inside.

![Discharge Tube Diagram](image)

It will discharge when the potential difference across the electrodes is higher than a certain threshold. When an electric field is applied across the tube, electrons and positive ions accelerate, but are soon slowed by collisions. But, if the field is sufficiently high, the electrons and ions will have enough energy on collision to ionize the atoms they collide with, so a cascading discharge will occur. The potential difference rises exponentially on an \((RC)\) time-scale until it reaches the threshold value, and the neon tube suddenly discharges. Then it starts all over again.

There is a similar problem involving an inductor in Chapter 10, Section 10.12.

**Integrating and differentiating circuits**

We look now at what happens if we connect a resistor and a capacitor in series across a voltage source that is varying with time, and we shall show that, provided some conditions are satisfied, the potential difference across the capacitor is the time integral of the input voltage, while the potential difference across the resistor is the time derivative of the input voltage.

We have seen that, if we connect a resistor and a capacitor in series with a battery of EMF \((V)\), the charge in the capacitor will increase according to

\[
Q = CV \left(1 - e^{-\frac{t}{RC}}\right)
\]

and asymptotically approaching \((Q=CV)\), and reaching \((1-e^{-1}=0.632)\) of this value in time \((RC)\). Note that, when \((t<<RC)\), the current will be large, and the charge in the capacitor will be small. Most of the potential drop in the circuit will be across the resistor, and relatively little across the capacitor. After a long time, however, the current will be low, and the charge will be high, so that most of the potential drop will be across the capacitor, and relatively little across the resistor. The potential drops across \(R\) and \(C\) will be equal at a time

\[
t = RC \ln 2 = 0.693RC
\]

Suppose that, instead of connecting \((R)\) and \((C)\) to a battery of constant EMF, we connect it to a source whose voltage varies with time, \((V(t))\). How will the charge in \((C)\) vary with time?
The relevant Equation is $V = IR + \frac{Q}{C}$, in which $I$, $Q$ and $V$ are all functions of time.

Since $I = \dot{Q}$, the differential Equation showing how $Q$ varies with time is

$$\frac{dQ}{dt} + \frac{1}{RC}Q = \frac{V}{R} \label{5.19.14}$$

The integration of this Equation is made easy if we multiply both sides by $e^{\frac{t}{RC}}$. (Those who are experienced in solving differential Equations will readily think of this step. Those who are less experienced might not immediately think of it, but will soon see that it is a useful step.) We then obtain

$$e^{\frac{t}{RC}}\frac{dQ}{dt} + \frac{1}{RC}e^{\frac{t}{RC}}Q = \frac{d}{dt}(Qe^{\frac{t}{RC}}) = \frac{V}{R}e^{\frac{t}{RC}} \label{5.19.15}$$

Thus the answer to our question is

$$Q = \frac{V}{R}e^{\frac{t}{RC}} \left( 1 - e^{-\frac{t}{RC}} \right) \label{5.19.16}$$

If $V$ is independent of time, this reduces to the familiar $Q = CV\left(1 - e^{-\frac{t}{RC}}\right)$.

The potential difference across $C$ increases, of course, as

$$V_c = \frac{1}{RC} \int V \, dt \label{5.19.18}$$

While $t$ is very much shorter than the time constant $RC$, by which I mean short enough that $e^{\frac{-t}{RC}}$ is very close to 1, this becomes

$$\int V_c \, dt \left( 1 - e^{-\frac{t}{RC}} \right) \label{5.19.19}$$

That is why this circuit is called an *integrating circuit*. The output voltage across $C$ is $1/(RC)$ times the time integral of the input voltage $V$. This is also true if the input voltage is a periodic function of time with a period that is very much shorter than the time constant.

By way of example, suppose that $V = at^2$. If we put this in the right hand side of Equation \ref{5.19.17} and integrate, with initial condition $V_C = 0$ when $t = 0$, (do it!), we obtain

$$V_c = aR^2C^2\left( t^2 - \frac{2t}{RC} + 2 - 2e^{-\frac{t}{RC}} \right) \label{5.19.19}$$

For example, suppose the input voltage varies as $V = 5t^2$ volts, where $t$ is in seconds. If $R = 500 \Omega$ and $C = 400 \mu\text{F}$, what will be the potential difference across the capacitor after 0.3 s? We immediately see that $RC =
0.2 \text{s} and \( \frac{t}{RC} = 1.5 \). Substitute SI numbers in Equation \ref{5.19.19} to obtain \( V_C = 0.161 \text{ V} \).

If I write \( \frac{V_c}{aR^2C^2} \) and \( \frac{t}{RC} \) Equation \ref{5.19.19} in dimensionless form becomes

\[ y = x^2 - 2x + 2 - 2e^{-x} \tag{5.19.20} \]

If you Taylor expand this as far as \( x^3 \) (do it!), you get \( y = \frac{1}{3}x^3 \), which is just what you would get by using Equation \ref{5.19.18}, the Equation which is an approximation for a time that is short compared with \( \frac{t}{RC} \). The approximation is good as long as \( \left( \frac{t}{RC} \right)^4 \) is negligible. I show Equation \ref{5.19.20} and \( y = \frac{1}{3}x^3 \) in the graph below, in which \( V_C \) is in units of \( (aR^2C^2) \) and \( T \) is in units of \( (RC) \).

Equation \ref{5.19.17} (or, for short time intervals, Equation \ref{5.19.18}) gives us the voltage across \( C \) as a function of time. What about the voltage across \( R \)? That is evidently

\[ V_R = V - \frac{e^{-\frac{t}{RC}}}{RC} \int Ve^{\frac{t}{RC}} \, dt. \tag{5.19.21} \]

Differentiate with respect to time:

\[ \begin{align} \frac{dV_R}{dt} &= \frac{dV}{dt} - \frac{1}{RC} \left( e^{-\frac{t}{RC}} Ve^{\frac{t}{RC}} - \frac{e^{-\frac{t}{RC}}}{RC} \int Ve^{\frac{t}{RC}} \, dt \right) \\
&= \frac{dV}{dt} - \frac{V_R}{RC} \end{align} \tag{5.19.22} \]

If the time constant is small so that \( \frac{dV_R}{dt} \ll \frac{V_R}{RC} \), this becomes

\[ V_R = RC \frac{dV}{dt}, \tag{5.20.23} \]

so that the voltage across \( R \) is \( RC \) times the time derivative of the input voltage \( V \). Thus we have a differentiating circuit.

Note that, in the integrating circuit, the circuit must have a large time constant (large \( R \) and \( C \)) and time variations in \( V \) are rapid compared with \( RC \). The output voltage across \( C \) is then \( \frac{1}{RC} \int V \, dt \). In the differentiating
circuit, the circuit must have a small time constant, and time variations in $V$ are slow compared with $(RC)$. The output voltage across $R$ is then $\frac{dV}{dR}$.

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