11.1: Simple Harmonic Motion

I am assuming that this is by no means the first occasion on which the reader has met simple harmonic motion, and hence in this section I merely summarize the familiar formulas without spending time on numerous elementary examples.

Simple harmonic motion can be defined as follows: If a point P moves in a circle of radius $a$ at constant angular speed $\omega$ (and hence period $\frac{2\pi}{\omega}$) its projection Q on a diameter moves with simple harmonic motion. This is illustrated in Figure XI.1, in which the velocity and acceleration of P and of Q are shown as coloured arrows. The velocity of P is just $a\omega$ and its acceleration is the centripetal acceleration $a\omega^2$. As in Chapter 8 and elsewhere, I use blue arrows for velocity vectors and green for acceleration.
\( P_0 \) is the initial position of \( P \) - i.e. the position of \( P \) at time \( t=0 \) - and \( a \) is the initial phase angle. At time \( t \) later, the phase angle is \( \omega t + \alpha \). The projection of \( P \) upon a diameter is \( Q \). The displacement of \( Q \) from the origin, and its velocity and acceleration, are

\[
\begin{align*}
\text{y} &= a \sin(\omega t + \alpha) \quad \text{(11.1.1)} \\
\text{v} &= \omega a \cos(\omega t + \alpha) \quad \text{(11.1.2)} \\
\ddot{\text{y}} &= -a \omega^2 \sin(\omega t + \alpha) \quad \text{(11.1.3)}
\end{align*}
\]

Equations \( \text{(11.1.2)} \) and \( \text{(11.1.3)} \) can be obtained immediately either by inspection of Figure XI.1 or by differentiation of Equation \( \text{(11.1.1)} \). Elimination of the time from Equations \( \text{(11.1.1)} \) and \( \text{(11.1.2)} \) and from Equations \( \text{(11.1.1)} \) and \( \text{(11.1.3)} \) leads to

\[
\begin{align*}
\text{v} &= \omega (a^2 - y^2)^{\frac{1}{2}} \quad \text{(11.1.4)} \\
\ddot{\text{y}} &= -\omega^2 y \quad \text{(11.1.5)}
\end{align*}
\]

An alternative definition of simple harmonic motion is to define as simple harmonic motion any motion that obeys the differential Equation \( \text{(11.1.5)} \). We then have the problem of solving this differential Equation. We can make no progress with this unless we remember to write \( \ddot{\text{y}} = \omega (a^2 - y^2)^{\frac{1}{2}} \) (recall that we did this often in Chapter 6.) Equation \( \text{(11.1.5)} \) then immediately integrates to

\[
\begin{align*}
\text{v}^2 &= \omega^2 (a^2 - y^2) \quad \text{(11.1.4)} \\
\ddot{\text{y}} &= -\omega^2 y \quad \text{(11.1.5)}
\end{align*}
\]

A further integration, with \( \text{v} = \frac{dy}{dt} \), leads to
$y = a \sin (\omega t + \alpha)$

provided we remember to use the appropriate initial conditions. Differentiation with respect to time then leads to Equation \ref{11.1.2}, and all the other Equations follow.

Exercise \PageIndex{1}

*Important Problem.*

Show that $y = a \sin (\omega t + \alpha)$ can be written

$$y = A \sin \omega t + B \cos \omega t \label{11.1.8}$$

where $A = a \cos \alpha$ and $B = a \sin \alpha$. The converse of these are $a = \sqrt{A^2 + B^2}$, $\cos \alpha = \frac{A}{\sqrt{A^2 + B^2}}$, $\sin \alpha = \frac{B}{\sqrt{A^2 + B^2}}$. It is important to note that, if $(A)$ and $(B)$ are known, in order to calculate $\alpha$ without ambiguity of quadrant it is entirely necessary to calculate both $(\cos \alpha)$ and $(\sin \alpha)$. It will not do, for example, to calculate $(\alpha)$ solely from $(\alpha = \tan^{-1} (\frac{y}{x}))$ because this will give two possible solutions for $\alpha$ differing by $180^\circ$.

Show also that Equation \ref{11.1.8} can also be written

$$y = Me^{i\omega t} + Ne^{-i\omega t}, \label{11.1.9}$$

where $M = \frac{1}{2}(B-iA)$ and $N = \frac{1}{2}(B+iA)$ show that the right hand side of Equation \ref{11.1.9} is real.

The four large satellites of Jupiter furnish a beautiful demonstration of simple harmonic motion. Earth is almost in the plane of their orbits, so we see the motion of satellites projected on a diameter. They move to and fro in simple harmonic motion, each with different amplitude (radius of the orbit), period (and hence angular speed $(\omega)$) and initial phase angle $(\alpha)$.

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