3.11: Torque and Rate of Change of Angular Momentum

Theorem:

The rate of change of the total angular momentum of a system of particles is equal to the sum of the external torques on the system.

Thus:

\[ L = \sum_i \mathbf{r}_i \times \mathbf{p}_i \tag{3.11.1} \label{eq:3.11.1} \]

\[ \therefore \dot{L} = \sum_i \dot{\mathbf{r}}_i \times \dot{\mathbf{p}}_i \tag{3.11.2} \label{eq:3.11.2} \]

But the first term is zero, because \( \dot{\mathbf{r}} \) and \( \mathbf{p}_i \) are parallel.

Also

\[ \dot{\mathbf{L}} = \sum_i \mathbf{r}_i \times (\mathbf{r}_i + \sum_j \mathbf{F}_{ij}) \]

\[ \therefore \quad \sum_i \mathbf{r}_i \times \mathbf{F}_i + \sum_i \mathbf{r}_i \times \sum_j \mathbf{F}_{ii} \]

But \( \sum_i \sum_j \mathbf{F}_{ij} = 0 \) by Newton’s third law of motion, and so \( \sum_i \sum_j \mathbf{r}_i \times \mathbf{F}_{ij} = 0 \).
Also \( \sum_i \{\textbf{r}_i \times \textbf{F}_i \} = \textbf{\tau} \), and so we arrive at

\[
\dot{\textbf{L}} = \textbf{\tau} \tag{3.11.4}\label{eq:3.11.4}
\]

which was to be demonstrated.

Corollary: Law of Conservation of Angular Momentum

If the sum of the external torques on a system is zero, the angular momentum is constant.

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