3.12: Torque, Angular Momentum and a Moving Point

In Figure III.7 I draw the particle $m_i$, which is just one of $n$ particles, of which I haven’t drawn and are scattered around in 3-space. I draw an arbitrary origin O, the centre of mass C of the system, and another point Q, which may (or may not) be moving with respect to O. The question I am going to ask is: Does the equation $\dot{\mathbf{L}} = \mathbf{\tau}$ apply to the point Q? It obviously does if Q is stationary, just as it applies to O. But what if Q is moving? If it does not apply, just what is the appropriate relation?

The theorem that we shall prove – and interpret - is

$$\dot{\mathbf{L}}_Q = \mathbf{\tau}_Q + M \mathbf{r}_Q' \times \ddot{\mathbf{r}}_Q.$$  \hfill (3.12.1)

We start:
\[ \dot{\mathbf{L}}_{\text{Q}} = \sum (\mathbf{r}_i - \mathbf{r}_\text{Q}) \times m_i (\mathbf{v}_i - \mathbf{v}_\text{Q}) \] \label{eq:3.12.2} \]

The second term is zero, because \( \dot{\mathbf{r}} = \mathbf{v} \)

Continue:

\[ \dot{\mathbf{L}} = \sum (\mathbf{r}_i - \mathbf{r}_\text{Q}) \times m_i \dot{\mathbf{v}}_i - \sum m_i \mathbf{r}_i \times \dot{\mathbf{v}}_\text{Q} + \sum m_i \mathbf{r}_\text{Q} \times \dot{\mathbf{v}}_\text{Q} \] \label{eq:3.12.4} \]

Now \( m_i \dot{\mathbf{v}}_i = \mathbf{F}_i \), so that the first term is just \( \mathbf{\tau}_\text{Q} \)

Continue:

\[ \dot{\mathbf{L}} = \mathbf{\tau}_\text{Q} - \sum m_i \mathbf{r}_i \times \dot{\mathbf{v}}_\text{Q} + \sum M_i \mathbf{r}_\text{Q} \times \dot{\mathbf{v}}_\text{Q} \]

\[ = \mathbf{\tau}_\text{Q} + M (\mathbf{r}_\text{Q} - \overline{\mathbf{r}}) \times \ddot{\mathbf{r}}_\text{Q} \]

\[ \therefore \dot{\mathbf{L}}_{\text{Q}} = \mathbf{\tau}_\text{Q} + M \mathbf{r}_\text{Q}' \times \ddot{\mathbf{r}}_\text{Q} \quad \text{Q.E.D.} \] \label{eq:3.12.5} \]

Thus in general, \( \dot{\mathbf{L}}_{\text{Q}} \neq \mathbf{\tau}_\text{Q} \), but \( \dot{\mathbf{L}}_{\text{Q}} = \mathbf{\tau}_\text{Q} \) under any of the following three circumstances:

i. \( (\mathbf{r}_\text{Q}' \times (\mathbf{r}_\text{Q}) = 0) \) - that is, \( \mathbf{Q} \) coincides with \( \mathbf{C} \).

ii. \( (\dot{\mathbf{r}}_\text{Q} \times (\mathbf{r}_\text{Q}) = 0) \) - that is, \( \mathbf{Q} \) is not accelerating.

iii. \( (\dot{\mathbf{r}}_\text{Q} \times (\mathbf{r}_\text{Q}') = 0) \) - that is, \( \mathbf{Q} \) is not accelerating, and \( \mathbf{Q} \) is parallel, which would happen, for example, if \( \mathbf{O} \) were a centre of attraction or repulsion and \( \mathbf{Q} \) were accelerating towards or away from \( \mathbf{O} \).