0.2 Scaling and Order-of-Magnitude Estimates

0.2.1 Introduction

Why can't an insect be the size of a dog? Some skinny stretched-out cells in your spinal cord are a meter tall --- why does nature display no single cells that are not just a meter tall, but a meter wide, and a meter thick as well? Believe it or not, these are questions that can be answered fairly easily without knowing much more about physics than you already do. The only mathematical technique you really need is the humble conversion, applied to area and volume.

Area and volume

Area can be defined by saying that we can copy the shape of interest onto graph paper with 1 cm \(\times\) 1 cm squares and count the number of squares inside. Fractions of squares can be estimated by eye. We then say the area equals the number of squares, in units of square cm. Although this might seem less “pure” than computing areas using formulae like \(A=\pi r^2\)
for a circle or \(A=\pi r^2\) for a triangle, those formulae are not useful as definitions of area because they cannot be applied to irregularly shaped areas.

Units of square cm are more commonly written as \(\text{cm}^2\) in science. Of course, the unit of measurement symbolized by “cm” is not an algebra symbol standing for a number that can be literally multiplied by itself. But it is advantageous to write the units of area that way and treat the units as if they were algebra symbols. For instance, if you have a rectangle with an area of \(6 \text{m}^2\) and a width of 2 m, then calculating its length as \(\frac{(6 \text{m}^2)}{(2 \text{m})}=3 \text{m}\) gives a result that makes sense both numerically and in terms of units. This algebra-style treatment of the units also ensures that our methods of converting units work out correctly. For instance, if we accept the fraction

\[
\frac{100 \text{cm}}{1 \text{m}}
\]

as a valid way of writing the number one, then one times one equals one, so we should also say that one can be represented by

\[
\frac{100 \text{cm}}{1 \text{m}} \times \frac{100 \text{cm}}{1 \text{m}} = \frac{10000 \text{cm}^2}{1 \text{m}^2}
\]

which is the same as

\[
\frac{10000 \text{cm}^2}{1 \text{m}^2}
\]

That means the conversion factor from square meters to square centimeters is a factor of \(10^4\), i.e., a square meter has \(10^4\) square centimeters in it.

All of the above can be easily applied to volume as well, using one-cubic-centimeter blocks instead of squares on graph paper.

To many people, it seems hard to believe that a square meter equals 10000 square centimeters, or that a cubic meter equals a million cubic centimeters --- they think it would make more sense if there were \(100 \text{cm}^2\) in \(1 \text{m}^2\), and \(100 \text{cm}^3\) in \(1 \text{m}^3\), but that would be incorrect. The examples shown in figure b aim to make the correct answer more believable, using the traditional U.S. units of feet and yards. (One foot is 12 inches, and one yard is three feet.)

\[
1 \text{ ft} \quad 1 \text{ yd} = 3 \text{ ft}
\]

\[
1 \text{ in}^2 = 9 \text{ in}^2
\]

\[
1 \text{ in}^3 = 1 \text{ yd}^3 = 27 \text{ in}^3
\]

b / Visualizing conversions of area and volume using traditional U.S. units.

self-check:

Based on figure b, convince yourself that there are 9 \(\text{in}^2\) in a square yard, and 27 \(\text{in}^3\) in a cubic yard, then demonstrate the same thing symbolically (i.e., with the method using fractions that equal one).

(answer in the back of the PDF version of the book)

◊ Solved problem: converting \(\text{mm}^2\) to \(\text{cm}^2\) — problem 31
Discussion Question

◊ How many square centimeters are there in a square inch? (1 inch = 2.54 cm) First find an approximate answer by making a drawing, then derive the conversion factor more accurately using the symbolic method.”

Great fleas have lesser fleas

Upon their backs to bite ’em.
And lesser fleas have lesser still,
And so ad infinitum. -- Jonathan Swift

Now how do these conversions of area and volume relate to the questions I posed about sizes of living things? Well, imagine that you are shrunk like Alice in Wonderland to the size of an insect. One way of thinking about the change of scale is that what used to look like a centimeter now looks like perhaps a meter to you, because you're so much smaller. If area and volume scaled according to most people's intuitive, incorrect expectations, with \((1 \text{ m}^2)\) being the same as 100 \((1 \text{ cm}^2)\), then there would be no particular reason why nature should behave any differently on your new, reduced scale. But nature does behave differently now that you're small. For instance, you will find that you can walk on water, and jump to many times your own height. The physicist Galileo Galilei had the basic insight that the scaling of area and volume determines how natural phenomena behave differently on different scales. He first reasoned about mechanical structures, but later extended his insights to living things, taking the then-radical point of view that at the fundamental level, a living organism should follow the same laws of nature as a machine. We will follow his lead by first discussing machines and then living things.
Galileo on the behavior of nature on large and small scales

One of the world's most famous pieces of scientific writing is Galileo's Dialogues Concerning the Two New Sciences. Galileo was an entertaining writer who wanted to explain things clearly to laypeople, and he livened up his work by casting it in the form of a dialogue among three people. Salviati is really Galileo's alter ego. Simplicio is the stupid character, and one of the reasons Galileo got in trouble with the Church was that there were rumors that Simplicio represented the Pope. Sagredo is the earnest and intelligent student, with whom the reader is supposed to identify. (The following excerpts are from the 1914 translation by Crew and de Salvio.)

Sagredo:

Yes, that is what I mean; and I refer especially to his last assertion which I have always regarded as false...; namely, that in speaking of these and other similar machines one cannot argue from the small to the large, because many devices which succeed on a small scale do not work on a large scale. Now, since mechanics has its foundations in geometry, where mere size [is unimportant], I do not see that the properties of circles, triangles, cylinders, cones and other solid figures will change with their size. If, therefore, a large machine be constructed in such a way that its parts bear to one another the same ratio as in a smaller one, and if the smaller is sufficiently strong for the purpose for which it is designed, I do not see why the larger should not be able to withstand any severe and destructive tests to which it may be subjected.

Salviati contradicts Sagredo:

Salviati:

... Please observe, gentlemen, how facts which at first seem improbable will, even on scant explanation, drop the cloak which has hidden them and stand forth in naked and simple beauty. Who does not know that a horse falling from a height of three or four cubits will break his bones, while a dog falling from the same height or a cat from a height of eight or ten cubits will suffer no injury? Equally harmless would be the fall of a grasshopper from a tower or the fall of an ant from the distance of the moon.

The point Galileo is making here is that small things are sturdier in proportion to their size. There are a lot of objections that could be raised, however. After all, what does it really mean for something to be “strong”, to be “strong in proportion to its size,” or to be strong “out of proportion to its size?” Galileo hasn't given operational definitions of things like “strength,” i.e., definitions that spell out how to measure them numerically.

Also, a cat is shaped differently from a horse --- an enlarged photograph of a cat would not be mistaken for a horse, even if the photo-doctoring experts at the National Inquirer made it look like a person was riding on its back. A grasshopper is not even a mammal, and it has an exoskeleton instead of an internal skeleton. The whole argument would be a lot more convincing if we could do some isolation of variables, a scientific term that means to change only one thing at a time, isolating it from the other variables that might have an effect. If size is the variable whose effect we're interested in seeing, then we don't really want to compare things that are different in size but also different in other ways.

Salviati:
... we asked the reason why [shipbuilders] employed stocks, scaffolding, and bracing of larger dimensions for launching a big vessel than they do for a small one; and [an old man] answered that they did this in order to avoid the danger of the ship parting under its own heavy weight, a danger to which small boats are not subject?

![Diagram of a small boat](image)

*d / The small boat holds up just fine.*

![Diagram of a larger boat](image)

*e / A larger boat built with the same proportions as the small one will collapse under its own weight.*

![Diagram of two planks](image)

*f / A boat this large needs to have timbers that are thicker compared to its size.*

After this entertaining but not scientifically rigorous beginning, Galileo starts to do something worthwhile by modern standards. He simplifies everything by considering the strength of a wooden plank. The variables involved can then be narrowed down to the type of wood, the width, the thickness, and the length. He also gives an operational definition of what it means for the plank to have a certain strength “in proportion to its size,” by introducing the concept of a plank that is the longest one that would not snap under its own weight if supported at one end. If you increased its length by the slightest amount, without increasing its width or thickness, it would break. He says that if one plank is the same shape as another but a different size, appearing like a reduced or enlarged photograph of the other, then the planks would be strong “in proportion to their sizes” if both were just barely able to support their own weight.
g / 1. This plank is as long as it can be without collapsing under its own weight. If it was a hundredth of an inch longer, it would collapse. 2. This plank is made out of the same kind of wood. It is twice as thick, twice as long, and twice as wide. It will collapse under its own weight.

h / Galileo discusses planks made of wood, but the concept may be easier to imagine with clay. All three clay rods in the figure were originally the same shape. The medium-sized one was twice the height, twice the length, and twice the width of the small one, and similarly the large one was twice as big as the medium one in all its linear dimensions. The big one has four times the linear dimensions of the small one, 16 times the cross-sectional area when cut perpendicular to the page, and 64 times the volume. That means that the big one has 64 times the weight to support, but only 16 times the strength compared to the smallest one.

Also, Galileo is doing something that would be frowned on in modern science: he is mixing experiments whose results he has actually observed (building boats of different sizes), with experiments that he could not possibly have done (dropping an ant from the height of the moon). He now relates how he has done actual experiments with such planks, and found that, according to this operational definition, they are not strong in proportion to their sizes. The larger one breaks. He makes sure to tell the reader how important the result is, via Sagredo's astonished response:

Sagredo:

My brain already reels. My mind, like a cloud momentarily illuminated by a lightning flash, is for an instant filled with an unusual light, which now beckons to me and which now suddenly mingles and obscures strange, crude ideas. From what you
have said it appears to me impossible to build two similar structures of the same material, but of different sizes and have them proportionately strong.

In other words, this specific experiment, using things like wooden planks that have no intrinsic scientific interest, has very wide implications because it points out a general principle, that nature acts differently on different scales.

To finish the discussion, Galileo gives an explanation. He says that the strength of a plank (defined as, say, the weight of the heaviest boulder you could put on the end without breaking it) is proportional to its cross-sectional area, that is, the surface area of the fresh wood that would be exposed if you sawed through it in the middle. Its weight, however, is proportional to its volume.²

How do the volume and cross-sectional area of the longer plank compare with those of the shorter plank? We have already seen, while discussing conversions of the units of area and volume, that these quantities don't act the way most people naively expect. You might think that the volume and area of the longer plank would both be doubled compared to the shorter plank, so they would increase in proportion to each other, and the longer plank would be equally able to support its weight. You would be wrong, but Galileo knows that this is a common misconception, so he has Salviati address the point specifically:

Salviati:

... Take, for example, a cube two inches on a side so that each face has an area of four square inches and the total area, i.e., the sum of the six faces, amounts to twenty-four square inches; now imagine this cube to be sawed through three times [with cuts in three perpendicular planes] so as to divide it into eight smaller cubes, each one inch on the side, each face one inch square, and the total surface of each cube six square inches instead of twenty-four in the case of the larger cube. It is evident therefore, that the surface of the little cube is only one-fourth that of the larger, namely, the ratio of six to twenty-four; but the volume of the solid cube itself is only one-eighth; the volume, and hence also the weight, diminishes therefore much more rapidly than the surface... You see, therefore, Simplicio, that I was not mistaken when ... I said that the surface of a small solid is comparatively greater than that of a large one.

The same reasoning applies to the planks. Even though they are not cubes, the large one could be sawed into eight small ones, each with half the length, half the thickness, and half the width. The small plank, therefore, has more surface area in proportion to its weight, and is therefore able to support its own weight while the large one breaks.

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Scaling of area and volume for irregularly shaped objects

You probably are not going to believe Galileo's claim that this has deep implications for all of nature unless you can be convinced that the same is true for any shape. Every drawing you've seen so far has been of squares, rectangles, and rectangular solids. Clearly the reasoning about sawing things up into smaller pieces would not prove anything about, say, an egg, which cannot be cut up into eight smaller egg-shaped objects with half the length.

Is it always true that something half the size has one quarter the surface area and one eighth the volume, even if it has an irregular shape? Take the example of a child's violin. Violins are made for small children in smaller size to accommodate their small bodies. Figure i shows a full-size violin, along with two violins made with half and 3/4 of the normal length.³ Let's
study the surface area of the front panels of the three violins.

\[ \text{i} \hspace{1em} \text{The area of a shape is proportional to the square of its linear dimensions, even if the shape is irregular.} \]

Consider the square in the interior of the panel of the full-size violin. In the 3/4-size violin, its height and width are both smaller by a factor of 3/4, so the area of the corresponding, smaller square becomes \((3/4 \times 3/4 = 9/16)\) of the original area,
not 3/4 of the original area. Similarly, the corresponding square on the smallest violin has half the height and half the width of the original one, so its area is 1/4 the original area, not half.

The same reasoning works for parts of the panel near the edge, such as the part that only partially fills in the other square. The entire square scales down the same as a square in the interior, and in each violin the same fraction (about 70%) of the square is full, so the contribution of this part to the total area scales down just the same.

Since any small square region or any small region covering part of a square scales down like a square object, the entire surface area of an irregularly shaped object changes in the same manner as the surface area of a square: scaling it down by 3/4 reduces the area by a factor of 9/16, and so on.

In general, we can see that any time there are two objects with the same shape, but different linear dimensions (i.e., one looks like a reduced photo of the other), the ratio of their areas equals the ratio of the squares of their linear dimensions:

\[
\frac{A_1}{A_2} = \left(\frac{L_1}{L_2}\right)^2.
\]

Note that it doesn't matter where we choose to measure the linear size, \(L\), of an object. In the case of the violins, for instance, it could have been measured vertically, horizontally, diagonally, or even from the bottom of the left f-hole to the middle of the right f-hole. We just have to measure it in a consistent way on each violin. Since all the parts are assumed to shrink or expand in the same manner, the ratio \(L_1/L_2\) is independent of the choice of measurement.

It is also important to realize that it is completely unnecessary to have a formula for the area of a violin. It is only possible to derive simple formulas for the areas of certain shapes like circles, rectangles, triangles and so on, but that is no impediment to the type of reasoning we are using.

Sometimes it is inconvenient to write all the equations in terms of ratios, especially when more than two objects are being compared. A more compact way of rewriting the previous equation is

\[
A \propto L^2.
\]

The symbol “\(\propto\)” means “is proportional to.” Scientists and engineers often speak about such relationships verbally using the phrases “scales like” or “goes like,” for instance “area goes like length squared.”

All of the above reasoning works just as well in the case of volume. Volume goes like length cubed:

\[
V \propto L^3.
\]

**self-check:**

When a car or truck travels over a road, there is wear and tear on the road surface, which incurs a cost. Studies show that the cost \(C\) per kilometer of travel is related to the weight per axle \(w\) by \(C \propto w^4\). Translate this into a statement about ratios.

(answer in the back of the PDF version of the book)

If different objects are made of the same material with the same density, \(\rho = m/V\), then their masses, \(m = \rho V\), are
proportional to \(L^3\). (The symbol for density is \(\rho\), the lower-case Greek letter “rho.”)

An important point is that all of the above reasoning about scaling only applies to objects that are the same shape. For instance, a piece of paper is larger than a pencil, but has a much greater surface-to-volume ratio.

\(\text{j / The muffin comes out of the oven too hot to eat. Breaking it up into four pieces increases its surface area while keeping the total volume the same. It cools faster because of the greater surface-to-volume ratio. In general, smaller things have greater surface-to-volume ratios, but in this example there is no easy way to compute the effect exactly, because the small pieces aren't the same shape as the original muffin.}\)

**Example 5: Scaling of the area of a triangle**

\(\text{k / Example 5. The big triangle has four times more area than the little one.}\)

\(\triangleright \text{In figure k, the larger triangle has sides twice as long. How many times greater is its area?}\)

Correct solution #1: Area scales in proportion to the square of the linear dimensions, so the larger triangle has four times more area \((2^2=4)\).

Correct solution #2: You could cut the larger triangle into four of the smaller size, as shown in fig. (b), so its area is four times greater. (This solution is correct, but it would not work for a shape like a circle, which can't be cut up into smaller circles.)
Correct solution #3: The area of a triangle is given by

\[
(A=\frac{bh}{2}), \text{ where } (b) \text{ is the base and } (h) \text{ is the height. The areas of the triangles are}
\[
\begin{align*}
A_1 &= \frac{b_1 h_1}{2} \\
A_2 &= \frac{b_2 h_2}{2} \\
A_2 &= \frac{(2b_1)(2h_1)}{2} \\
A_2/A_1 &= \frac{2b_1 h_1}{\frac{b_1 h_1}{2}} \times 4
\end{align*}
\]

(Although this solution is correct, it is a lot more work than solution #1, and it can only be used in this case because a triangle is a simple geometric shape, and we happen to know a formula for its area.)

Correct solution #4: The area of a triangle is \(A=\frac{bh}{2}\). The comparison of the areas will come out the same as long as the ratios of the linear sizes of the triangles is as specified, so let's just say \((b_1=1.00)\) m and \((b_2=2.00)\) m. The heights are then also \((h_1=1.00)\) m and \((h_2=2.00)\) m, giving areas \((A_1=0.50\text{ m}^2)\) and \((A_2=2.00\text{ m}^2)\), so \(A_2/A_1=4.00\).

(The solution is correct, but it wouldn't work with a shape for whose area we don't have a formula. Also, the numerical calculation might make the answer of 4.00 appear inexact, whereas solution #1 makes it clear that it is exactly 4.)

Incorrect solution: The area of a triangle is \(A=\frac{bh}{2}\), and if you plug in \((b=2.00)\) m and \((h=2.00)\) m, you get \((A=2.00\text{ m}^2)\), so the bigger triangle has 2.00 times more area. (This solution is incorrect because no comparison has been made with the smaller triangle.)

**Example 6: Scaling of the volume of a sphere**
Example 6. The big sphere has 125 times more volume than the little one.

In figure m, the larger sphere has a radius that is five times greater. How many times greater is its volume?

Correct solution #1: Volume scales like the third power of the linear size, so the larger sphere has a volume that is 125 times greater \(5^3\).

Correct solution #2: The volume of a sphere is \(V=(4/3)\pi r^3\), so
\[
\begin{align*}
V_1 &= \frac{4}{3}\pi r_1^3 \\
V_2 &= \frac{4}{3}\pi (5r_1)^3 \\
&= \frac{500}{3}\pi r_1^3 \\
V_2/V_1 &= \left( \frac{500}{3}\pi r_1^3 \right) / \left( \frac{4}{3}\pi r_1^3 \right) &= 125
\end{align*}
\]

Incorrect solution: The volume of a sphere is \(V=(4/3)\pi r^3\), so
\[
\begin{align*}
V_1 &= \frac{4}{3}\pi r_1^3 \\
V_2 &= \frac{4}{3}\pi 5r_1^3 \\
&= \frac{20}{3}\pi r_1^3 \\
V_2/V_1 &= \left( \frac{20}{3}\pi r_1^3 \right) / \left( \frac{4}{3}\pi r_1^3 \right) &= 5
\end{align*}
\]
(The solution is incorrect because \((5r_1)^3\) is not the same as \(5r_1^3\).)

Example 7: Scaling of a more complex shape

The 48-point “S” has 1.78 times more area than the 36-point “S.”

The first letter “S” in figure n is in a 36-point font, the second in 48-point. How many times more ink is required to make the larger “S”? (Points are a unit of length used in typography.)
Correct solution: The amount of ink depends on the area to be covered with ink, and area is proportional to the square of the linear dimensions, so the amount of ink required for the second “S” is greater by a factor of \((48/36)^2=1.78\).

Incorrect solution: The length of the curve of the second “S” is longer by a factor of \(48/36=1.33\), so 1.33 times more ink is required.

(The solution is wrong because it assumes incorrectly that the width of the curve is the same in both cases. Actually both the width and the length of the curve are greater by a factor of 48/36, so the area is greater by a factor of \((48/36)^2=1.78\).)

Reasoning about ratios and proportionalities is one of the three essential mathematical skills, summarized on pp.905-907, that you need for success in this course.

◊ Solved problem: a telescope gathers light — problem 32

◊ Solved problem: distance from an earthquake — problem 33

Discussion Questions

◊ A toy fire engine is 1/30 the size of the real one, but is constructed from the same metal with the same proportions. How many times smaller is its weight? How many times less red paint would be needed to paint it?

◊ Galileo spends a lot of time in his dialog discussing what really happens when things break. He discusses everything in terms of Aristotle's now-discredited explanation that things are hard to break, because if something breaks, there has to be a gap between the two halves with nothing in between, at least initially. Nature, according to Aristotle, “abhors a vacuum,” i.e., nature doesn't “like” empty space to exist. Of course, air will rush into the gap immediately, but at the very moment of breaking, Aristotle imagined a vacuum in the gap. Is Aristotle's explanation of why it is hard to break things an experimentally testable statement? If so, how could it be tested experimentally?

0.2.3 Order-of-magnitude estimates

It is the mark of an instructed mind to rest satisfied with the degree of precision that the nature of the subject permits and not to seek an exactness where only an approximation of the truth is possible. -- *Aristotle*
Can you guess how many jelly beans are in the jar? If you try to guess directly, you will almost certainly underestimate. The right way to do it is to estimate the linear dimensions, then get the volume indirectly. See problem 44, p. 53.

It is a common misconception that science must be exact. For instance, in the Star Trek TV series, it would often happen that Captain Kirk would ask Mr. Spock, “Spock, we're in a pretty bad situation. What do you think are our chances of getting out of here?” The scientific Mr. Spock would answer with something like, “Captain, I estimate the odds as 237.345 to one.” In reality, he could not have estimated the odds with six significant figures of accuracy, but nevertheless one of the hallmarks of a person with a good education in science is the ability to make estimates that are likely to be at least somewhere in the right ballpark. In many such situations, it is often only necessary to get an answer that is off by no more than a factor of ten in either direction. Since things that differ by a factor of ten are said to differ by one order of magnitude, such an estimate is called an order-of-magnitude estimate. The tilde, \( \sim \), is used to indicate that things are only of the same order of magnitude, but not exactly equal, as in

\[
\text{odds of survival} \sim \text{100 to one}.
\]

The tilde can also be used in front of an individual number to emphasize that the number is only of the right order of magnitude.

Although making order-of-magnitude estimates seems simple and natural to experienced scientists, it's a mode of reasoning that is completely unfamiliar to most college students. Some of the typical mental steps can be illustrated in the following example.

**Example 8: Cost of transporting tomatoes (incorrect solution)**

Roughly what percentage of the price of a tomato comes from the cost of transporting it in a truck?
The following incorrect solution illustrates one of the main ways you can go wrong in order-of-magnitude estimates.

Incorrect solution: Let's say the trucker needs to make a $400 profit on the trip. Taking into account her benefits, the cost of gas, and maintenance and payments on the truck, let's say the total cost is more like $2000. I'd guess about 5000 tomatoes would fit in the back of the truck, so the extra cost per tomato is 40 cents. That means the cost of transporting one tomato is comparable to the cost of the tomato itself. Transportation really adds a lot to the cost of produce, I guess.

The problem is that the human brain is not very good at estimating area or volume, so it turns out the estimate of 5000 tomatoes fitting in the truck is way off. That's why people have a hard time at those contests where you are supposed to estimate the number of jellybeans in a big jar. Another example is that most people think their families use about 10 gallons of water per day, but in reality the average is about 300 gallons per day. When estimating area or volume, you are much better off estimating linear dimensions, and computing volume from the linear dimensions.

Here's a better solution to the problem about the tomato truck:

**Example 9: Cost of transporting tomatoes (correct solution)**

As in the previous solution, say the cost of the trip is $2000. The dimensions of the bin are probably 4 m \( \times \) 2 m \( \times \) 1 m, for a volume of \( 8 \text{m}^3 \). Since the whole thing is just an order-of-magnitude estimate, let's round that off to the nearest power of ten, \( 10 \text{m}^3 \). The shape of a tomato is complicated, and I don't know any formula for the volume of a tomato shape, but since this is just an estimate, let's pretend that a tomato is a cube, 0.05 m \( \times \) 0.05 m \( \times \) 0.05 m, for a volume of \( 1.25 \times 10^{-4} \text{m}^3 \). Since this is just a rough estimate, let's round
that to $10^{-4}\text{m}^3$. We can find the total number of tomatoes by dividing the volume of the bin by the volume of one tomato: $10\text{m}^3/10^{-4}\text{m}^3=10^5$ tomatoes. The transportation cost per tomato is $2000/10^5$ tomatoes = $0.02/tomato. That means that transportation really doesn't contribute very much to the cost of a tomato.

Approximating the shape of a tomato as a cube is an example of another general strategy for making order-of-magnitude estimates. A similar situation would occur if you were trying to estimate how many $\text{m}^2$ of leather could be produced from a herd of ten thousand cattle. There is no point in trying to take into account the shape of the cows' bodies. A reasonable plan of attack might be to consider a spherical cow. Probably a cow has roughly the same surface area as a sphere with a radius of about 1 m, which would be $(4\pi (1\text{m})^2)$. Using the well-known facts that pi equals three, and four times three equals about ten, we can guess that a cow has a surface area of about $(10^2\text{m}^2)$, so the herd as a whole might yield $(10^5\times10^2\text{m}^2)$ of leather.

**Example 10: Estimating mass indirectly**

Usually the best way to estimate mass is to estimate linear dimensions, then use those to infer volume, and then get the mass based on the volume. For example, *Amphicoelias*, shown in the figure, may have been the largest land animal ever to live. Fossils tell us the linear dimensions of an animal, but we can only indirectly guess its mass. Given the length scale in the figure, let's estimate the mass of an *Amphicoelias*.

Its torso looks like it can be approximated by a rectangular box with dimensions $(10\text{m}\times5\text{m}\times3\text{m})$, giving about $(2\times10^2\text{m}^3)$. Living things are mostly made of water, so we assume the animal to have the density of water, $(1\text{g}/\text{cm}^3)$, which converts to $(10^3\text{kg}/\text{m}^3)$. This gives a mass of about $(2\times10^5\text{kg})$, or 200 metric tons.

The following list summarizes the strategies for getting a good order-of-magnitude estimate.

1. Don't even attempt more than one significant figure of precision.
2. Don't guess area, volume, or mass directly. Guess linear dimensions and get area, volume, or mass from them.
3. When dealing with areas or volumes of objects with complex shapes, idealize them as if they were some simpler shape, a cube or a sphere, for example.
4. Check your final answer to see if it is reasonable. If you estimate that a herd of ten thousand cattle would yield $(0.01\text{m}^2)$ of leather, then you have probably made a mistake with conversion factors somewhere.
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