4.3: Resonance

Resonance is a phenomenon in which an oscillator responds most strongly to a driving force that matches its own natural frequency of vibration. For example, suppose a child is on a playground swing with a natural frequency of 1 Hz. That is, if you pull the child away from equilibrium, release her, and then stop doing anything for a while, she'll oscillate at 1 Hz. If there was no friction, as we assumed in section 2.5, then the sum of her gravitational and kinetic energy would remain constant, and the amplitude would be exactly the same from one oscillation to the next. However, friction is going to convert these forms of energy into heat, so her oscillations would gradually die out. To keep this from happening, you might give her a push once per cycle, i.e., the frequency of your pushes would be 1 Hz, which is the same as the swing's natural frequency. As long as you stay in rhythm, the swing responds quite well. If you start the swing from rest, and then give pushes at 1 Hz, the swing's amplitude rapidly builds up, as in figure a, until after a while it reaches a steady state in which friction removes just as much energy as you put in over the course of one cycle.

**self-check:**

![Figure a: An x-versus-t graph for a swing pushed at resonance.](image)

In figure a, compare the amplitude of the cycle immediately following the first push to the amplitude after the second. Compare the energies as well. (answer in the back of the PDF version of the book)

What will happen if you try pushing at 2 Hz? Your first push puts in some momentum, \( \langle p \rangle \), but your second push happens after only half a cycle, when the swing is coming right back at you, with momentum \( \langle -p \rangle \)! The momentum transfer from the
second push is exactly enough to stop the swing. The result is a very weak, and not very sinusoidal, motion, b.

![Figure b: A swing pushed at twice its resonant frequency.](image)

**Making the math easy**

This is a simple and physically transparent example of resonance: the swing responds most strongly if you match its natural rhythm. However, it has some characteristics that are mathematically ugly and possibly unrealistic. The quick, hard pushes are known as impulse forces, c, and they lead to an \( \langle x \rangle - \langle t \rangle \) graph that has nondifferentiable kinks.

![Figure c: The \( \langle F \rangle \)-versus-\( \langle t \rangle \) graph for an impulsive driving force.](image)

Impulsive forces like this are not only badly behaved mathematically, they are usually undesirable in practical terms. In a car engine, for example, the engineers work very hard to make the force on the pistons change smoothly, to avoid excessive vibration. Throughout the rest of this section, we'll assume a driving force that is sinusoidal, d, i.e., one whose \( \langle F \rangle - \langle t \rangle \) graph is either a sine function or a function that differs from a sine wave in phase, such as a cosine. The force is positive for half of each cycle and negative for the other half, i.e., there is both pushing and pulling. Sinusoidal functions have many nice mathematical characteristics (we can differentiate and integrate them, and the sum of sinusoidal functions that have the same frequency is a sinusoidal function), and they are also used in many practical situations. For instance, my garage door zapper sends out a sinusoidal radio wave, and the receiver is tuned to resonance with it.

![Figure d: A sinusoidal driving force.](image)

A second mathematical issue that I glossed over in the swing example was how friction behaves. In section 3.2.4, about forces between solids, the empirical equation for kinetic friction was independent of velocity. Fluid friction, on the other hand, is velocity-dependent. For a child on a swing, fluid friction is the most important form of friction, and is approximately proportional to \( v^2 \). In still other situations, e.g., with a low-density gas or friction between solid surfaces that have been lubricated with a fluid such as oil, we may find that the frictional force has some other dependence on velocity, perhaps being proportional to \( v \), or having some other complicated velocity dependence that can't even be expressed with a simple equation. It would be extremely complicated to have to treat all of these different possibilities in complete generality, so for the rest of this section, we'll assume friction proportional to velocity.
\[ \begin{equation*} F = -bv , \end{equation*} \]
simply because the resulting equations happen to be the easiest to solve. Even when the friction doesn't behave in exactly this way, many of our results may still be at least qualitatively correct.

### 3.3.1 Damped, free motion

**Numerical treatment**

An oscillator that has friction is referred to as damped. Let's use numerical techniques to find the motion of a damped oscillator that is released away from equilibrium, but experiences no driving force after that. We can expect that the motion will consist of oscillations that gradually die out.

![Figure](image_url)

**Figure**: A damped sine wave, of the form \( x = Ae^{-ct}\sin(\omega_{f}t+\delta) \).

In section 2.5, we simulated the undamped case using our tried and true Python function based on conservation of energy. Now, however, that approach becomes a little awkward, because it involves splitting up the path to be traveled into \( n \) tiny segments, but in the presence of damping, each swing is a little shorter than the last one, and we don't know in advance exactly how far the oscillation will get before turning around. An easier technique here is to use force rather than energy. Newton's second law, \( (a=F/m) \), gives \( (a=(-kx-bv)/m) \), where we've made use of the result of example 40 for the force exerted by the spring. This becomes a little prettier if we rewrite it in the form

\[ \begin{equation*} ma+bv+kx = 0 , \end{equation*} \]

which gives symmetric treatment to three terms involving \( x \) and its first and second derivatives, \( v \) and \( a \). Now instead of calculating the time \( (\Delta{t}=\Delta{x}/v) \) required to move a predetermined distance \( (\Delta{x}) \), we pick \( (\Delta{t}) \) and determine the distance traveled in that time, \( (\Delta{x}=v\Delta{t}) \). Also, we can no longer update \( v \) based on conservation of energy, since we don't have any easy way to keep track of how much mechanical energy has been changed into heat energy. Instead, we recalculate the velocity using \( (v=a\Delta{t}) \).

```python
import math
k=39.4784 # chosen to give a period of 1 second
m=1.
b=0.211 # chosen to make the results simple
x=1.
v=0.
t=0.
```

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dt=.01
n=1000
for j in range(n):
    x=x+v*dt
    a=(-k*x-b*v)/m
    if (v>0) and (v+a*dt<0) :
        print("turnaround at t=",t,"  x=",x)
    v=v+a*dt
    t=t+dt

turnaround at t= 0.99 , x= 0.899919262445
turnaround at t= 1.99 , x= 0.809844934046
turnaround at t= 2.99 , x= 0.72877519477
turnaround at t= 3.99 , x= 0.655817260033
turnaround at t= 4.99 , x= 0.590154191135
turnaround at t= 5.99 , x= 0.531059189965
turnaround at t= 6.99 , x= 0.477875914756
turnaround at t= 7.99 , x= 0.430013546991
turnaround at t= 8.99 , x= 0.386940256644
turnaround at t= 9.99 , x= 0.348177318484

The spring constant, $(k=4\pi{}=39.4784) \text{ N/m}$, is designed so that if the undamped equation $(f=(1/2\pi\sqrt{k/m}))$ was still true, the frequency would be 1 Hz. We start by noting that the addition of a small amount of damping doesn't seem to have changed the period at all, or at least not to within the accuracy of the calculation. You can check for yourself, however, that a large value of $b$, say $5 \text{ N}\cdot\text{s}/\text{m}$, does change the period significantly.

We release the mass from $x=1 \text{ m}$, and after one cycle, it only comes back to about $x=0.9 \text{ m}$. I chose $b=0.211 \text{ N}\cdot\text{s}/\text{m}$ by fiddling around until I got this result, since a decrease of exactly 10% is easy to discuss. Notice how the amplitude after two cycles is about $(0.81 \text{ m})$, i.e., $(0.9 \text{ m})$ times $(0.9^2)$: the amplitude has again dropped by exactly 10%. This pattern continues for as long as the simulation runs, e.g., for the last two cycles, we have $0.34818/0.38694=0.89982$, or almost exactly 0.9 again. It might have seemed capricious when I chose to use the unrealistic equation $(F=-bv)$, but this is the payoff. Only with $(bv)$ friction do we get this kind of mathematically simple exponential decay.

Because the decay is exponential, it never dies out completely; this is different from the behavior we would have had with Coulomb friction, which does make objects grind completely to a stop at some point. With friction that acts like $(F=-bv)$, $(v)$ gets smaller as the oscillations get smaller. The smaller and smaller force then causes them to die out at a rate that is slower and slower.

### Analytic treatment

Taking advantage of this unexpectedly simple result, let's find an analytic solution for the motion. The numerical output suggests that we assume a solution of the form

\[
\begin{equation*}
x = Ae^{-ct}\sin (\omega f t+\delta),
\end{equation*}
\]
where the unknown constants $\omega_f$ and $c$ will presumably be related to $m$, $b$, and $k$. The constant $c$ indicates how quickly the oscillations die out. The constant $\omega_f$ is, as before, defined as $2\pi$ times the frequency, with the subscript $f$ to indicate a free (undriven) solution. All our equations will come out much simpler if we use $\omega$s everywhere instead of $f$s from now on, and, as physicists often do, I'll generally use the word “frequency” to refer to $\omega$s when the context makes it clear what I'm talking about. The phase angle $\delta$ has no real physical significance, since we can define $t=0$ to be any moment in time we like.

**self-check:**

In figure f, which graph has the greater value of $c$? (answer in the back of the PDF version of the book)

The factor $A$ for the initial amplitude can also be omitted without loss of generality, since the equation we're trying to solve, $ma+bv+kx = 0$, is linear. That is, $(v)$ and $(a)$ are the first and second derivatives of $(x)$, and the derivative of $(Ax)$ is simply $(A)$ times the derivative of $(x)$. Thus, if $(x(t))$ is a solution of the equation, then multiplying it by a constant gives an equally valid solution. This is another place where we see that a damping force proportional to $v$ is the easiest to handle mathematically. For a damping force proportional to $v^2$, for example, we would have had to solve the equation $ma+bv^2+kx = 0$, which is nonlinear.

For the purpose of determining $\omega_f$ and $c$, the most general form we need to consider is therefore $x = e^{-ct}\sin \omega_f t$, whose first and second derivatives are $(v = e^{-ct}\left(-c \sin \omega_f t + \omega \cos \omega_f t \right))$ and $(a = e^{-ct}\left(c^2 \sin \omega_f t - 2\omega c \cos \omega_f t - \omega_f^2 \sin \omega_f t \right))$. Plugging these into the equation $ma+bv+kx = 0$ and setting the sine and cosine parts equal to zero gives, after some tedious algebra,

\[
\begin{align*}
c &= \frac{b}{2m} \\
\omega_f &= \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}
\end{align*}
\]

Intuitively, we expect friction to “slow down” the motion, as when we ride a bike into a big patch of mud. “Slow down,” however, could have more than one meaning here. It could mean that the oscillator would take more time to complete each cycle, or it could mean that as time went on, the oscillations would die out, thus giving smaller velocities.

Our mathematical results show that both of these things happen. The first equation says that $c$, which indicates how quickly the oscillations damp out, is directly related to $b$, the strength of the damping.

The second equation, for the frequency, can be compared with the result from page 116 of $\sqrt{k/m}$ for the undamped system. Let's refer to this now as $\omega_f$, to distinguish it from the actual frequency $\omega_f$ of the free system.
oscillations when damping is present. The result for \( \omega_f \) will be less than \( \omega_o \), due to the presence of the \( b^2/4m^2 \) term. This tells us that the addition of friction to the system does increase the time required for each cycle. However, it is very common for the \( b^2/4m^2 \) term to be negligible, so that \( \omega_f \approx \omega_o \).

Figure g: A damped sine wave is compared with an undamped one, with \( m \) and \( k \) kept the same and only \( b \) changed.

Figure g shows an example. The damping here is quite strong: after only one cycle of oscillation, the amplitude has already been reduced by a factor of 2, corresponding to a factor of 4 in energy. However, the frequency of the damped oscillator is only about 1% lower than that of the undamped one; after five periods, the accumulated lag is just barely visible in the offsetting of the arrows. We can see that extremely strong damping --- even stronger than this --- would have been necessary in order to make \( \omega_f \approx \omega_o \) a poor approximation.

### 3.3.2 The quality factor

It's usually impractical to measure \( b \) directly and determine \( c \) from the equation \( c = b/2m \). For a child on a swing, measuring \( b \) would require putting the child in a wind tunnel! It's usually much easier to characterize the amount of damping by observing the actual damped oscillations and seeing how many cycles it takes for the mechanical energy to decrease by a certain factor. The unitless *quality factor*, \( Q \), is defined as \( Q = \omega_o / 2c \), and in the limit of weak damping, where \( \omega \approx \omega_o \), this can be interpreted as the number of cycles required for the mechanical energy to fall off by a factor of \( e^{2\pi} = 535.49... \). Using this new quantity, we can rewrite the equation for the frequency of damped oscillations in the slightly more elegant form \( \omega_f = \omega_o \sqrt{1 - 1/4Q^2} \).

**self-check:**

What if we wanted to make a simpler definition of \( Q \), as the number of oscillations required for the vibrations to die out completely, rather than the number required for the energy to fall off by this obscure factor? (answer in the back of the PDF version of the book)
Example 43: A graph

The damped motion in figure g has \( Q \approx 4.5 \), giving \( \sqrt{1-1/4Q^2} \approx 0.99 \), as claimed at the end of the preceding subsection.

Example 44: Exponential decay in a trumpet

\( \triangleright \) The vibrations of the air column inside a trumpet have a \( Q \) of about 10. This means that even after the trumpet player stops blowing, the note will keep sounding for a short time. If the player suddenly stops blowing, how will the sound intensity 20 cycles later compare with the sound intensity while she was still blowing?

\( \triangleright \) The trumpet's \( Q \) is 10, so after 10 cycles the energy will have fallen off by a factor of 535. After another 10 cycles we lose another factor of 535, so the sound intensity is reduced by a factor of \( 535 \times 535 = 2.9 \times 10^5 \).

The decay of a musical sound is part of what gives it its character, and a good musical instrument should have the right \( Q \), but the \( Q \) that is considered desirable is different for different instruments. A guitar is meant to keep on sounding for a long time after a string has been plucked, and might have a \( Q \) of 1000 or 10000. One of the reasons why a cheap synthesizer sounds so bad is that the sound suddenly cuts off after a key is released.

3.3.3 Driven motion

**Summary of Notation**

- \( k \): spring constant
- \( m \): mass of the oscillator
- \( b \): sets the amount of damping, \( F = -bv \)
- \( T \): period
- \( f \): frequency, \( 1 / T \)
- \( \Omega \): (Greek letter omega), angular frequency, \( 2\pi f \), often referred to simply as “frequency”
- \( \omega_f \): frequency the oscillator would have without damping, \( \sqrt{k/m} \) frequency of the free vibrations sets the time scale for the exponential decay envelope \( e^{-ct} \) of the free vibrations
- \( F_m \): strength of the driving force, which is assumed to vary sinusoidally with frequency \( \omega \)
- \( A \): amplitude of the steady-state response
- \( \delta \): phase angle of the steady-state response

The driven case is extremely important in science, technology, and engineering. We have an external driving force \( F=F_m \)
\( \sin \omega t \), where the constant \( F_m \) indicates the maximum strength of the force in either direction. The equation of motion is now

\[
ma + bv + kx = F_m \sin \omega t \quad \text{[equation of motion for a driven oscillator]}
\]

After the driving force has been applied for a while, we expect that the amplitude of the oscillations will approach some constant value. This motion is known as the **steady state**, and it's the most interesting thing to find out; as we'll see later, the most general type of motion is only a minor variation on the steady-state motion. For the steady-state motion, we're going to look for a solution of the form

\[
x = A \sin (\omega t + \delta) \quad \text{[equation]}\]

In contrast to the undriven case, here it's not possible to sweep \( A \) and \( \delta \) under the rug. The amplitude of the steady-state motion, \( A \), is actually the most interesting thing to know about the steady-state motion, and it's not true that we still have a solution no matter how we fiddle with \( A \); if we have a solution for a certain value of \( A \), then multiplying \( A \) by some constant would break the equality between the two sides of the equation of motion. It's also no longer true that we can get rid of \( \delta \) simply by redefining when we start the clock; here \( \delta \) represents a *difference* in time between the start of one cycle of the driving force and the start of the corresponding cycle of the motion.

The velocity and acceleration are \( v = \omega A \cos (\omega t + \delta) \) and \( a = -\omega^2 A \sin (\omega t + \delta) \), and if we plug these into the equation of motion, \eqref{eqn:resonancemotion}, and simplify a little, we find

\[
(k-m\omega^2) \sin (\omega t + \delta) + \omega b \cos (\omega t + \delta) = \frac{F_m}{A} \sin \omega t
\]

The sum of any two sinusoidal functions with the same frequency is also a sinusoidal, so the whole left side adds up to a sinusoidal. By fiddling with \( A \) and \( \delta \) we can make the amplitudes and phases of the two sides of the equation match up.

**Steady state, no damping**

\( A \) and \( \delta \) are easy to find in the case where there is no damping at all. There are now no cosines in equation above, only sines, so if we wish we can set \( \delta \) to zero, and we find \( A = F_m/(k-m\omega^2) = F_m/m(\omega_o^2-\omega^2) \). This, however, makes \( A \) negative for \( \omega > \omega_o \). The variable \( \delta \) was designed to represent this kind of phase relationship, so we prefer to keep \( A \) positive and set \( \delta = \pi \) for \( \omega > \omega_o \). Our results are then

\[
A = \frac{F_m}{m(\omega_o^2-\omega^2)} \quad \text{and} \quad \delta = \pi \quad \text{for} \quad \omega > \omega_o
\]

The most important feature of the result is that there is a resonance: the amplitude becomes greater and greater, and approaches infinity, as \( \omega \) approaches the resonant frequency \( \omega_o \). This is the physical behavior we anticipated on page 171 in the example of pushing a child on a swing. If the driving frequency matches the frequency of the free vibrations, then the driving force will always be in the right direction to add energy to the swing. At a driving frequency
very different from the resonant frequency, we might get lucky and push at the right time during one cycle, but our next push would come at some random point in the next cycle, possibly having the effect of slowing the swing down rather than speeding it up.

The interpretation of the infinite amplitude at \(\omega=\omega_0\) is that there really isn't any steady state if we drive the system exactly at resonance --- the amplitude will just keep on increasing indefinitely. In real life, the amplitude can't be infinite both because there is always some damping and because there will always be some difference, however small, between \(\omega\) and \(\omega_0\). Even though the infinity is unphysical, it has entered into the popular consciousness, starting with the eccentric Serbian-American inventor and physicist Nikola Tesla. Around 1912, the tabloid newspaper *The World Today* credulously reported a story which Tesla probably fabricated --- or wildly exaggerated --- for the sake of publicity. Supposedly he created a steam-powered device “no larger than an alarm clock,” containing a piston that could be made to vibrate at a tunable and precisely controlled frequency. “He put his little vibrator in his coat-pocket and went out to hunt a half-erected steel building. Down in the Wall Street district, he found one --- ten stories of steel framework without a brick or a stone laid around it. He clamped the vibrator to one of the beams, and fussed with the adjustment [presumably hunting for the building's resonant frequency] until he got it. Tesla said finally the structure began to creak and weave and the steel-workers came to the ground panic-stricken, believing that there had been an earthquake. Police were called out. Tesla put the vibrator in his pocket and went away. Ten minutes more and he could have laid the building in the street. And, with the same vibrator he could have dropped the Brooklyn Bridge into the East River in less than an hour.”

The phase angle \(\delta\) also exhibits surprising behavior. As the frequency is tuned upward past resonance, the phase abruptly shifts so that the phase of the response is opposite to that of the driving force. There is a simple interpretation for this. The system's mechanical energy can only change due to work done by the driving force, since there is no damping to convert mechanical energy to heat. In the steady state, then, the power transmitted by the driving force over a full cycle of motion must average out to zero. In general, the work theorem \((dE=Fdx)\) can always be divided by \(d\tau\) on both sides to give the useful relation \((P=Fv)\). If \((Fv)\) is to average out to zero, then \((F)\) and \((v)\) must be out of phase by \(\pm\pi/2\), and since \((v)\) is ahead of \((x)\) by a phase angle of \(\pi/2\), the phase angle between \((x)\) and \((F)\) must be zero or \(\pi\).
Figure h: Dependence of the amplitude and phase angle on the driving frequency, for an undamped oscillator. The amplitudes were calculated with \( F_m \), \( m \), and \( \omega_0 \), all set to 1.

Given that these are the two possible phases, why is there a difference in behavior between \( \omega < \omega_0 \) and \( \omega > \omega_0 \)? At the low-frequency limit, consider \( \omega = 0 \), i.e., a constant force. A constant force will simply displace the oscillator to one side, reaching an equilibrium that is offset from the usual one. The force and the response are in phase, e.g., if the force is to the right, the equilibrium will be offset to the right. This is the situation depicted in the amplitude graph of figure h at \( \omega = 0 \). The response, which is not zero, is simply this static displacement of the oscillator to one side.

At high frequencies, on the other hand, imagine shaking the poor child on the swing back and forth with a force that oscillates at 10 Hz. This is so fast that there is essentially no time for the force \( F = -kx \) from gravity and the chain to act from one cycle to the next. The problem becomes equivalent to the oscillation of a free object. If the driving force varies like \( \sin(\omega t) \), with \( \delta = 0 \), then the acceleration is also proportional to the sine. Integrating, we find that the velocity goes like minus a cosine, and a second integration gives a position that varies as minus the sine --- opposite in phase to the driving force. Intuitively, this mathematical result corresponds to the fact that at the moment when the object has reached its maximum displacement to the right, that is the time when the greatest force is being applied to the left, in order to turn it around and bring it back toward the center.

**Example 45: A practice mute for a violin**

The amplitude of the driven vibrations, \( A = \frac{F_m}{m(\omega^2 - \omega_0^2)} \), contains an inverse proportionality to the mass of the vibrating object. This is simply because a given force will produce less acceleration when applied to a more massive object. An application is shown in figure 45.

![Example 45: a viola without a mute (left), and with a mute (right). The mute doesn't touch the strings themselves.](image)

In a stringed instrument, the strings themselves don't have enough surface area to excite sound waves very efficiently. In instruments of the violin family, as the strings vibrate from left to right, they cause the bridge (the piece of wood they pass over) to wiggle clockwise and counterclockwise, and this motion is transmitted to the top panel of the instrument, which vibrates and creates sound waves in the air.

A string player who wants to practice at night without bothering the neighbors can add some mass to the bridge. Adding mass to the bridge causes the amplitude of the vibrations to be smaller, and the sound to be much softer. A similar effect is seen when an electric guitar is used without an amp. The body of an electric guitar is so much more massive than the body
of an acoustic guitar that the amplitude of its vibrations is very small.

Steady state, with damping

The extension of the analysis to the damped case involves some lengthy algebra, which I've outlined on page 912 in appendix 2. The results are shown in figure j. It's not surprising that the steady state response is weaker when there is more damping, since the steady state is reached when the power extracted by damping matches the power input by the driving force. The maximum amplitude, at the peak of the resonance curve, is approximately proportional to \(Q\).

Figure j: Dependence of the amplitude and phase angle on the driving frequency. The undamped case is \(Q=\infty\), and the other curves represent \(Q\)=1, 3, and 10. \(F_m\), \(m\), and \(\omega_o\) are all set to 1.

**self-check:**

From the final result of the analysis on page 912, substitute \(\omega=\omega_o\), and satisfy yourself that the result is proportional to \(Q\). Why is \(A_{res}\propto Q\) only an approximation? (answer in the back of the PDF version of the book)
The definition of $\Delta \omega$, the full width at half maximum.

What is surprising is that the amplitude is strongly affected by damping close to resonance, but only weakly affected far from it. In other words, the shape of the resonance curve is broader with more damping, and even if we were to scale up a high-damping curve so that its maximum was the same as that of a low-damping curve, it would still have a different shape. The standard way of describing the shape numerically is to give the quantity $\Delta \omega$, called the full width at half-maximum, or FWHM, which is defined in figure k. Note that the $y$ axis is energy, which is proportional to the square of the amplitude. Our previous observations amount to a statement that $\Delta \omega$ is greater when the damping is stronger, i.e., when the $Q$ is lower. It's not hard to show from the equations on page 912 that for large $Q$, the FWHM is given approximately by

$$\Delta \omega \approx \omega_0/Q.$$  

Another thing we notice in figure j is that for small values of $Q$ the frequency $\omega_{res}$ of the maximum $A$ is less than $\omega_0$. At even lower values of $Q$, like $Q=1$, the $A-\omega$ curve doesn't even have a maximum near $\omega=0$.

**Example 46: An opera singer breaking a wineglass**

In order to break a wineglass by singing, an opera singer must first tap the glass to find its natural frequency of vibration, and then sing the same note back, so that her driving force will produce a response with the greatest possible amplitude. If she's shopping for the right glass to use for this display of her prowess, she should look for one that has the greatest possible $Q$, since the resonance curve has a higher maximum for higher values of $Q$.

In order to break a wineglass by singing, an opera singer must first tap the glass to find its natural frequency of vibration, and then sing the same note back, so that her driving force will produce a response with the greatest possible amplitude. If she's shopping for the right glass to use for this display of her prowess, she should look for one that has the greatest possible $Q$, since the resonance curve has a higher maximum for higher values of $Q$.

**Example 47: Collapse of the Nimitz Freeway**

Figure l shows a section of the Nimitz Freeway in Oakland, CA, that collapsed during an earthquake in 1989. An earthquake consists of many low-frequency vibrations that occur simultaneously, which is why it sounds like a rumble of indeterminate pitch rather than a low hum. The frequencies that we can hear are not even the strongest ones; most of the energy is in the form of vibrations in the range of frequencies from about 1 Hz to 10 Hz.
All the structures we build are resting on geological layers of dirt, mud, sand, or rock. When an earthquake wave comes along, the topmost layer acts like a system with a certain natural frequency of vibration, sort of like a cube of jello on a plate being shaken from side to side. The resonant frequency of the layer depends on how stiff it is and also on how deep it is. The ill-fated section of the Nimitz freeway was built on a layer of mud, and analysis by geologist Susan E. Hough of the U.S. Geological Survey shows that the mud layer’s resonance was centered on about 2.5 Hz, and had a width covering a range from about 1 Hz to 4 Hz.

When the earthquake wave came along with its mixture of frequencies, the mud responded strongly to those that were close to its own natural 2.5 Hz frequency. Unfortunately, an engineering analysis after the quake showed that the overpass itself had a resonant frequency of 2.5 Hz as well! The mud responded strongly to the earthquake waves with frequencies close to 2.5 Hz, and the bridge responded strongly to the 2.5 Hz vibrations of the mud, causing sections of it to collapse.

### Physical reason for the relationship between Q and the FWHM

What is the reason for this surprising relationship between the damping and the width of the resonance? Fundamentally, it has to do with the fact that friction causes a system to lose its “memory” of its previous state. If the Pioneer 10 space probe, coasting through the frictionless vacuum of interplanetary space, is detected by aliens a million years from now, they will be able to trace its trajectory backwards and infer that it came from our solar system. On the other hand, imagine that I shove a book along a tabletop, it comes to rest, and then someone else walks into the room. There will be no clue as to which direction the book was moving before it stopped --- friction has erased its memory of its motion.

![Graph of steady-state motion](image)

Figure m: An \(\langle x \rangle\)-versus-\(\langle t \rangle\) graph of the steady-state motion of a swing being pushed at twice its resonant frequency by an
impulsive force.

Now consider the playground swing driven at twice its natural frequency, figure m, where the undamped case is repeated from figure b on page 171. In the undamped case, the first push starts the swing moving with momentum \( p \), but when the second push comes, if there is no friction at all, it now has a momentum of exactly \( -p \), and the momentum transfer from the second push is exactly enough to stop it dead. With moderate damping, however, the momentum on the rebound is not quite \( -p \), and the second push's effect isn't quite as disastrous. With very strong damping, the swing comes essentially to rest long before the second push. It has lost all its memory, and the second push puts energy into the system rather than taking it out. Although the detailed mathematical results with this kind of impulsive driving force are different, the general results are the same as for sinusoidal driving: the less damping there is, the greater the penalty you pay for driving the system off of resonance.

**Example 48: High-Q speakers**

Most good audio speakers have \( Q \approx 1 \), but the resonance curve for a higher-\( Q \) oscillator always lies above the corresponding curve for one with a lower \( Q \), so people who want their car stereos to be able to rattle the windows of the neighboring cars will often choose speakers that have a high \( Q \). Of course they could just use speakers with stronger driving magnets to increase \( F_m \), but the speakers might be more expensive, and a high-\( Q \) speaker also has less friction, so it wastes less energy as heat.

One problem with this is that whereas the resonance curve of a low-\( Q \) speaker (its “response curve” or “frequency response” in audiophile lingo) is fairly flat, a higher-\( Q \) speaker tends to emphasize the frequencies that are close to its natural resonance. In audio, a flat response curve gives more realistic reproduction of sound, so a higher quality factor, \( Q \), really corresponds to a lower-quality speaker.

Another problem with high-\( Q \) speakers is discussed in example 51 on page 185.

**Example 49: Changing the pitch of a wind instrument**

\[
\text{A saxophone player normally selects which note to play by choosing a certain fingering, which gives the saxophone a certain resonant frequency. The musician can also, however, change the pitch significantly by altering the tightness of her lips. This corresponds to driving the horn slightly off of resonance. If the pitch can be altered by about 5% up or down (about one musical half-step) without too much effort, roughly what is the \( Q \) of a saxophone?}
\]

\[
\text{Five percent is the width on one side of the resonance, so the full width is about 10%, \( \delta f / f \approx 0.1 \). The equation \( \omega = \omega_0 / Q \) is defined in terms of angular frequency, \( \omega = 2\pi f \), and we've been given our data in terms of ordinary frequency, \( f \). The factors of \( 2\pi \) end up canceling out, however:}
\]

\[
\text{\begin{align*}
Q &= \frac{\omega_0}{\Delta \omega} \\
&= \frac{2\pi f_0}{2\pi \Delta f} \\
&= \frac{f_0}{f} \\
&\approx 10
\end{align*}}
\]
In other words, once the musician stops blowing, the horn will continue sounding for about 10 cycles before its energy falls off by a factor of 535. (Blues and jazz saxophone players will typically choose a mouthpiece that gives a low \(Q\), so that they can produce the bluesy pitch-slides typical of their style. “Legit,” i.e., classically oriented players, use a higher-\(Q\) setup because their style only calls for enough pitch variation to produce a vibrato, and the higher \(Q\) makes it easier to play in tune.)

Example 50: Q of a radio receiver

A radio receiver used in the FM band needs to be tuned in to within about 0.1 MHz for signals at about 100 MHz. What is its \(Q\)?

As in the last example, we're given data in terms of \(f\)'s, not \(\omega\)'s, but the factors of \(2\pi\) cancel. The resulting \(Q\) is about 1000, which is extremely high compared to the \(Q\) values of most mechanical systems.

Transients

What about the motion before the steady state is achieved? When we computed the undriven motion numerically on page 172, the program had to initialize the position and velocity. By changing these two variables, we could have gotten any of an infinite number of simulations. The same is true when we have an equation of motion with a driving term, \(ma+bv+kx = F_m \sin \omega t\). The steady-state solutions, however, have no adjustable parameters at all --- \(A\) and \(\delta\) are uniquely determined by the parameters of the driving force and the oscillator itself. If the oscillator isn't initially in the steady state, then it will not have the steady-state motion at first. What kind of motion will it have?

The answer comes from realizing that if we start with the solution to the driven equation of motion, and then add to it any solution to the free equation of motion, the result,

\[
x = A \sin (\omega t+\delta) + A' e^{-ct} \sin (\omega_f t+\delta'),
\]

is also a solution of the driven equation. Here, as before, \(\omega_f\) is the frequency of the free oscillations (\(\omega_f \approx \omega_0\) for small \(Q\)), \(\omega\) is the frequency of the driving force, \(A\) and \(\delta\) are related as usual to the parameters of the driving force, and \(A'\) and \(\delta'\) can have any values at all. Given the initial position and velocity, we can always choose \(A'\) and \(\delta'\) to reproduce them, but this is not something one often has to do in real life. What's more important is to realize that the second term dies out exponentially over time, decaying at the same rate at which a free vibration would. For this reason, the \(A'\) term is called a transient. A high-\(Q\) oscillator's transients take a long time to die out, while a low-\(Q\) oscillator always settles down to its steady state very quickly.
Example 51: Boomy bass

In example 48 on page 183, I’ve already discussed one of the drawbacks of a high-$Q$ speaker, which is an uneven response curve. Another problem is that in a high-$Q$ speaker, transients take a long time to die out. The bleeding-ear-drum crowd tend to focus mostly on making their bass loud, so it's usually their woofers that have high $Q$'s. The result is that bass notes, “ring” after the onset of the note, a phenomenon referred to as “boomy bass.”

Overdamped motion

The treatment of free, damped motion on page 174 skipped over a subtle point: in the equation $\omega_f = \sqrt{k/m-b^2/4m^2}$, $\sqrt{1-1/4Q^2}$ results in an answer that is the square root of a negative number. For example, suppose we had $(k=0)$, which corresponds to a neutral equilibrium. A physical example would be a mass sitting in a tub of syrup. If we set it in motion, it won't oscillate --- it will simply slow to a stop. This system has $(Q=0)$. The equation of motion in this case is $(ma+bv=0)$, or, more suggestively,

$$m\frac{dv}{dt}+bv=0.$$

One can easily verify that this has the solution $(v=(\text{constant})e^{-bt/m})$, and integrating, we find $(x=(\text{constant})e^{-bt/m}+(\text{constant}))$. In other words, the reason $\omega_f$ comes out to be mathematical nonsense is that we were incorrect in assuming a solution that oscillated at a frequency $\omega_f$. The actual motion is not oscillatory at all.

In general, systems with $Q<1/2$, called overdamped systems, do not display oscillatory motion. Most cars' shock absorbers are designed with $Q\approx1/2$, since it's undesirable for the car to undulate up and down for a while after you go over a bump. (Shocks with extremely low values of $Q$ are not good either, because such a system takes a very long time to come back to equilibrium.) It's not particularly important for our purposes, but for completeness I'll note, as you can easily verify, that the general solution to the equation of motion for $0<Q<1/2$ is of the form $(x=Ae^{-ct}+Be^{-dt})$, while $(Q=1/2)$, called the critically damped case, gives $(x=(A+Bt)e^{-ct})$.

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