4.4: Motion In Three Dimensions

3.4.1 The Cartesian perspective

When my friends and I were bored in high school, we used to play a paper-and-pencil game which, although we never knew it, was Very Educational --- in fact, it pretty much embodies the entire world-view of classical physics. To play the game, you draw a racetrack on graph paper, and try to get your car around the track before anyone else. The default is for your car to continue at constant speed in a straight line, so if it moved three squares to the right and one square up on your last turn, it will do the same this turn. You can also control the car's motion by changing its \(\Delta x\) and \(\Delta y\) by up to one unit. If it moved three squares to the right last turn, you can have it move anywhere from two to four squares to the right this turn.

![Diagram](image)

a / The car can change its \(\Delta x\) and \(\Delta y\) motions by one square every turn.
French mathematician René Descartes invented analytic geometry; Cartesian (xyz) coordinates are named after him. He did work in philosophy, and was particularly interested in the mind-body problem. He was a skeptic and an anti-Aristotelian, and, probably for fear of religious persecution, spent his adult life in the Netherlands, where he fathered a daughter with a Protestant peasant whom he could not marry. He kept his daughter's existence secret from his enemies in France to avoid giving them ammunition, but he was crushed when she died of scarlatina at age 5. A pious Catholic, he was widely expected to be sainted. His body was buried in Sweden but then reburied several times in France, and along the way everything but a few fingerbones was stolen by peasants who expected the body parts to become holy relics.

The fundamental way of dealing with the direction of an object's motion in physics is to use conservation of momentum, since momentum depends on direction. Up until now, we've only done momentum in one dimension. How does this relate to the racetrack game? In the game, the motion of a car from one turn to the next is represented by its \( \Delta x \) and \( \Delta y \). In one dimension, we would only need \( \Delta x \), which could be related to the velocity, \( \Delta x / \Delta t \), and the momentum, \( m \Delta x / \Delta t \). In two dimensions, the rules of the game amount to a statement that if there is no momentum transfer, then both \( m \Delta x / \Delta t \) and \( m \Delta y / \Delta t \) stay the same. In other words, there are two flavors of momentum, and they are separately conserved. All of this so far has been done with an artificial division of time into “turns,” but we can fix that by redefining everything in terms of derivatives, and for motion in three dimensions rather
than two, we augment \((x')\) and \((y')\) with \((z')\):

\[
\begin{align*}
  v_x &= \frac{dx}{dt} & v_y &= \frac{dy}{dt} & v_z &= \frac{dz}{dt} \\
  p_x &= mv_x & p_y &= mv_y & p_z &= mv_z
\end{align*}
\]

We call these the \((x')\), \((y')\), and \((z')\) components of the velocity and the momentum.

There is both experimental and theoretical evidence that the \((x')\), \((y')\), and \((z')\) momentum components are separately conserved, and that a momentum transfer (force) along one axis has no effect on the momentum components along the other two axes. On page 89, for example, I argued that it was impossible for an air hockey puck to make a 180-degree turn spontaneously, because then in the frame moving along with the puck, it would have begun moving after starting from rest. Now that we're working in two dimensions, we might wonder whether the puck could spontaneously make a 90-degree turn, but exactly the same line of reasoning shows that this would be impossible as well, which proves that the puck can't trade \((x')\)-momentum for \((y')\)-momentum. A more general proof of separate conservation will be given on page 214, after some of the appropriate mathematical techniques have been introduced.

As an example of the experimental evidence for separate conservation of the momentum components, figure c shows correct and incorrect predictions of what happens if you shoot a rifle and arrange for a second bullet to be dropped from the same height at exactly the same moment when the first one left the barrel. Nearly everyone expects that the dropped bullet will reach the dirt first, and Aristotle would have agreed, since he believed that the bullet had to lose its horizontal motion before it could start moving vertically. In reality, we find that the vertical momentum transfer between the earth and the bullet is completely unrelated to the horizontal momentum. The bullet ends up with \((p_y<0)\), while the planet picks up an upward momentum \((p_y>0)\), and the total momentum in the \((y')\) direction remains zero. Both bullets hit the ground at the same time. This is much simpler than the Aristotelian version!

**Example 52: The Pelton waterwheel**

There is a general class of machines that either do work on a gas or liquid, like a boat's propeller, or have work done on them by a gas or liquid, like the turbine in a hydroelectric power plant. Figure d shows two types of surfaces that could be attached to the circumference of an old-fashioned waterwheel. Compare the force exerted by the water in the two cases.
Figure d: Two surfaces that could be used to extract energy from a stream of water.

**SOLUTION**

Let the \((x)\) axis point to the right, and the \((y)\) axis up. In both cases, the stream of water rushes down onto the surface with momentum \(p_{y,i}=-p_o\), where the subscript \((i)\) stands for “initial,” i.e., before the collision.

In the case of surface 1, the streams of water leaving the surface have no momentum in the \((y)\) direction, and their momenta in the \((x)\) direction cancel. The final momentum of the water is zero along both axes, so its entire momentum, \((-p_o)\), has been transferred to the waterwheel.

When the water leaves surface 2, however, its momentum isn't zero. If we assume there is no friction, its \(p_{y,f}=+p_o\), with the positive sign indicating upward momentum. The change in the water's momentum is \((p_{y,f}-p_{y,i}=2p_o)\), and the momentum transferred to the waterwheel is \((-2p_o)\).

Force is defined as the rate of transfer of momentum, so surface 2 experiences double the force. A waterwheel constructed in this way is known as a Pelton waterwheel.

**Example 53: The Yarkovsky effect**

We think of the planets and asteroids as inhabiting their orbits permanently, but it is possible for an orbit to change over
periods of millions or billions of years, due to a variety of effects. For asteroids with diameters of a few meters or less, an important mechanism is the Yarkovsky effect, which is easiest to understand if we consider an asteroid spinning about an axis that is exactly perpendicular to its orbital plane.

The illuminated side of the asteroid is relatively hot, and radiates more infrared light than the dark (night) side. Light has momentum, and a total force away from the sun is produced by combined effect of the sunlight hitting the asteroid and the imbalance between the momentum radiated away on the two sides. This force, however, doesn't cause the asteroid's orbit to change over time, since it simply cancels a tiny fraction of the sun's gravitational attraction. The result is merely a tiny, undetectable violation of Kepler's law of periods.

![Diagram of an asteroid](image)

*Figure e: An asteroid absorbs visible light from the sun, and gets rid of the energy by radiating infrared light.*

Consider the sideways momentum transfers, however. In figure e, the part of the asteroid on the right has been illuminated for half a spin-period (half a “day”) by the sun, and is hot. It radiates more light than the morning side on the left. This imbalance produces a total force in the \( (x) \) direction which points to the left. If the asteroid's orbital motion is to the left, then this is a force in the same direction as the motion, which will do positive work, increasing the asteroid's energy and boosting it into an orbit with a greater radius. On the other hand, if the asteroid's spin and orbital motion are in opposite directions, the Yarkovsky push brings the asteroid spiraling in closer to the sun.

Calculations show that it takes on the order of \( 10^7 \) to \( 10^8 \) years for the Yarkovsky effect to move an asteroid out of the asteroid belt and into the vicinity of earth's orbit, and this is about the same as the typical age of a meteorite as estimated by its exposure to cosmic rays. The Yarkovsky effect doesn't remove all the asteroids from the asteroid belt, because many of them have orbits that are stabilized by gravitational interactions with Jupiter. However, when collisions occur, the fragments can end up in orbits which are not stabilized in this way, and they may then end up reaching the earth due to the Yarkovsky effect. The cosmic-ray technique is really telling us how long it has been since the fragment was broken out of its parent.
Discussion Questions

◊ The following is an incorrect explanation of a fact about target shooting:

“Shooting a high-powered rifle with a high muzzle velocity is different from shooting a less powerful gun. With a less powerful gun, you have to aim quite a bit above your target, but with a more powerful one you don't have to aim so high because the bullet doesn't drop as fast.”

What is the correct explanation?

◊ You have thrown a rock, and it is flying through the air in an arc. If the earth's gravitational force on it is always straight down, why doesn't it just go straight down once it leaves your hand?

◊ Consider the example of the bullet that is dropped at the same moment another bullet is fired from a gun. What would the motion of the two bullets look like to a jet pilot flying alongside in the same direction as the shot bullet and at the same horizontal speed?

3.4.2 Rotational invariance

The Cartesian approach requires that we choose \( \langle x \rangle, \langle y \rangle, \) and \( \langle z \rangle \) axes. How do we choose them correctly? The answer is that it had better not matter which directions the axes point (provided they're perpendicular to each other), or where we put the origin, because if it did matter, it would mean that space was asymmetric. If there was a certain point in the universe that was the right place to put the origin, where would it be? The top of Mount Olympus? The United Nations headquarters? We find that experiments come out the same no matter where we do them, and regardless of which way the laboratory is oriented, which indicates that no location in space or direction in space is special in any way.\(^{15}\)
Two balls roll down a cone and onto a plane.

This is closely related to the idea of Galilean relativity stated on page 62, from which we already know that the absolute motion of a frame of reference is irrelevant and undetectable. Observers using frames of reference that are in motion relative to each other will not even agree on the permanent identity of a particular point in space, so it's not possible for the laws of physics to depend on where you are in space. For instance, if gravitational energies were proportional to \((m_1m_2)\) in one location but to \((m_1m_2)^{1.00001}\) in another, then it would be possible to determine when you were in a state of absolute motion, because the behavior of gravitational interactions would change as you moved from one region to the other.

Because of this close relationship, we restate the principle of Galilean relativity in a more general form. This extended principle of Galilean relativity states that the laws of physics are no different in one time and place than in another, and that they also don’t depend on your orientation or your motion, provided that your motion is in a straight line and at constant speed.

The irrelevance of time and place could have been stated in chapter 1, but since this section is the first one in which we’re dealing with three-dimensional physics in full generality, the irrelevance of orientation is what we really care about right now. This property of the laws of physics is called rotational invariance. The word “invariance” means a lack of change, i.e., the laws of physics don’t change when we reorient our frame of reference.

**Example 54: Rotational invariance of gravitational interactions**

Gravitational energies depend on the quantity \(1/\langle r \rangle\), which by the Pythagorean theorem equals

\[
\|\begin{equation*}
\frac{1}{\sqrt{\Delta x^2+\Delta y^2+\Delta z^2}} . \|\end{equation*}\]

Rotating a line segment doesn’t change its length, so this expression comes out the same regardless of which way we orient our coordinate axes. Even though \(\langle\Delta x \rangle\), \(\langle\Delta y \rangle\), and \(\langle\Delta z \rangle\) are different in differently oriented coordinate systems, \(\langle r \rangle\) is the same.

**Example 55: Kinetic energy**

Kinetic energy equals \((1/2) mv^2\), but what does that mean in three dimensions, where we have \(\langle v_x \rangle\), \(\langle v_y \rangle\), and \(\langle v_z \rangle\)? If you were tempted to add the components and calculate \((K=(1/2) m( v_x+ v_y+ v_z)^2)\), figure g should convince you otherwise. Using that method, we’d have to assign a kinetic energy of zero to ball number 1, since its negative \(\langle v_y \rangle\) would exactly cancel its positive \(\langle v_x \rangle\), whereas ball number 2’s kinetic energy wouldn’t be zero. This would violate rotational invariance, since the balls would behave differently.

The only possible way to generalize kinetic energy to three dimensions, without violating rotational invariance, is to use an expression that resembles the Pythagorean theorem,

\[
\|\begin{equation*}
v=\sqrt{ v_x^2+ v_y^2+ v_z^2} . \|\end{equation*}\]

which results in
Since the velocity components are squared, the positive and negative signs don’t matter, and the two balls in the example behave the same way.

3.4.3 Vectors

Remember the title of this book? It would have been possible to obtain the result of example 55 by applying the Pythagorean theorem to \( \langle dx \rangle \), \( \langle dy \rangle \), and \( \langle dz \rangle \), and then dividing by \( \langle dt \rangle \), but the rotational invariance approach is simpler, and is useful in a much broader context. Even with a quantity you presently know nothing about, say the magnetic field, you can infer that if the components of the magnetic field are \( \langle B_x \rangle \), \( \langle B_y \rangle \), and \( \langle B_z \rangle \), then the physically useful way to talk about the strength of the magnetic field is to define it as \( \langle \sqrt{B_x^2+B_y^2+B_z^2} \rangle \). Nature knows your brain cells are precious, and doesn’t want you to have to waste them by memorizing mathematical rules that are different for magnetic fields than for velocities.

When mathematicians see that the same set of techniques is useful in many different contexts, that's when they start making definitions that allow them to stop reinventing the wheel. The ancient Greeks, for example, had no general concept of fractions. They couldn't say that a circle's radius divided by its diameter was equal to the number 1/2. They had to say that the radius and the diameter were in the ratio of one to two. With this limited number concept, they couldn't have said that water was dripping out of a tank at a rate of 3/4 of a barrel per day; instead, they would have had to say that over four days, three barrels worth of water would be lost. Once enough of these situations came up, some clever mathematician finally realized that it would make sense to define something called a fraction, and that one could think of these fraction thingies as numbers that lay in the gaps between the traditionally recognized numbers like zero, one, and two. Later generations of mathematicians introduced further subversive generalizations of the number concepts, inventing mathematical creatures like negative numbers, and the square root of two, which can't be expressed as a fraction.

In this spirit, we define a \textit{vector} as any quantity that has both an amount and a direction in space. In contradistinction, a \textit{scalar} has an amount, but no direction. Time and temperature are scalars. Velocity, acceleration, momentum, and force are vectors. In one dimension, there are only two possible directions, and we can use positive and negative numbers to indicate the two directions. In more than one dimension, there are infinitely many possible directions, so we can't use the two symbols \( (+) \) and \( (-) \) to indicate the direction of a vector. Instead, we can specify the three components of the vector, each of which can be either negative or positive. We represent vector quantities in handwriting by writing an arrow above them, so for example the momentum vector looks like this, \( \vec{p} \), but the arrow looks ugly in print, so in books vectors are usually shown in boldface type: \( \mathbf{p} \). A straightforward way of thinking about vectors is that a vector equation really represents three different equations. For instance, conservation of momentum could be written in terms of the three components, 

\[
\begin{align*}
\Delta p_x &= 0 \\
\Delta p_y &= 0 \\
\Delta p_z &= 0
\end{align*}
\]

or as a single vector equation, \(^{16}\)

\[
\Delta \mathbf{p} = 0
\]
The following table summarizes some vector operations.

<table>
<thead>
<tr>
<th>Operation</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>(</td>
<td>\mathbf{vector}</td>
</tr>
<tr>
<td>(\mathbf{vector}+\mathbf{vector})</td>
<td>Add component by component.</td>
</tr>
<tr>
<td>(\mathbf{vector}-\mathbf{vector})</td>
<td>Subtract component by component.</td>
</tr>
<tr>
<td>(\mathbf{vector}\cdot\text{scalar})</td>
<td>Multiply each component by the scalar.</td>
</tr>
<tr>
<td>(\mathbf{vector} / \text{scalar})</td>
<td>Divide each component by the scalar.</td>
</tr>
</tbody>
</table>

The first of these is called the *magnitude* of the vector; in one dimension, where a vector only has one component, it amounts to taking the absolute value, hence the similar notation.

**self-check:**

Translate the equations \(F_x = ma_x\), \(F_y = ma_y\), and \(F_z = ma_z\) into a single equation in vector notation.

(answer in the back of the PDF version of the book)

**Example 56: An explosion**

\(\triangleright\) Astronomers observe the planet Mars as the Martians fight a nuclear war. The Martian bombs are so powerful that they rip the planet into three separate pieces of liquefied rock, all having the same mass. If one fragment flies off with velocity components \(v_{1 x} = 0\), \(v_{1 y} = 1.0 \times 10^4\) km/hr, and the second with \(v_{2 x} = 1.0 \times 10^4\) km/hr, \(v_{2 y} = 0\), what is the magnitude of the third one’s velocity?
We work the problem in the center of mass frame, in which the planet initially had zero momentum. After the explosion, the vector sum of the momenta must still be zero. Vector addition can be done by adding components, so

\[
\begin{align*}
  mv_{1 x} + mv_{2 x} + mv_{3 x} &= 0 \\
  mv_{1 y} + mv_{2 y} + mv_{3 y} &= 0 ,
\end{align*}
\]

where we have used the same symbol \(m\) for all the terms, because the fragments all have the same mass. The masses can be eliminated by dividing each equation by \(m\), and we find

\[
\begin{align*}
  v_{3 x} &= -1.0 \times 10^4 \text{ km/hr} \\
  v_{3 y} &= -1.0 \times 10^4 \text{ km/hr} ,
\end{align*}
\]

which gives a magnitude of

\[
\begin{align*}
  |\mathbf{v}_3| &= \sqrt{ v_{3 x}^2 + v_{3 y}^2} \\
  &= 1.4 \times 10^4 \text{ km/hr} .
\end{align*}
\]

**Example 57: A toppling box**

If you place a box on a frictionless surface, it will fall over with a very complicated motion that is hard to predict in detail. We know, however, that its center of mass's motion is related to its momentum, and the rate at which momentum is transferred is the force. Moreover, we know that these relationships apply separately to each component. Let \(\langle x\rangle\) and \(\langle y\rangle\) be horizontal, and \(\langle z\rangle\) vertical. There are two forces on the box, an upward force from the table and a downward gravitational force. Since both of these are along the \(\langle z\rangle\) axis, \(\langle p_z\rangle\) is the only component of the box's momentum that can change. We conclude that the center of mass travels vertically. This is true even if the box bounces and tumbles. [Based on an example by Kleppner and Kolenkow.]
Geometric representation of vectors

A vector in two dimensions can be easily visualized by drawing an arrow whose length represents its magnitude and whose direction represents its direction. The \( \langle x \rangle \) component of a vector can then be visualized, \( j \), as the length of the shadow it would cast in a beam of light projected onto the \( \langle x \rangle \) axis, and similarly for the \( \langle y \rangle \) component. Shadows with arrowheads pointing back against the direction of the positive axis correspond to negative components.
In this type of diagram, the negative of a vector is the vector with the same magnitude but in the opposite direction. Multiplying a vector by a scalar is represented by lengthening the arrow by that factor, and similarly for division.
Two vectors, \( \mathbf{1} \), to which we apply the same operation in two different frames of reference, \( \mathbf{2} \) and \( \mathbf{3} \).

**Self-check:**

Given vector \( \mathbf{Q} \) represented by an arrow below, draw arrows representing the vectors \( 1.5 \mathbf{Q} \) and \( -\mathbf{Q} \).

(answer in the back of the PDF version of the book)

**A useless vector operation**

The way I've defined the various vector operations above aren't as arbitrary as they seem. There are many different vector operations that we could define, but only some of the possible definitions are mathematically useful. Consider the operation of multiplying two vectors component by component to produce a third vector:

\[
\begin{align*}
R_x &= P_x Q_x \\
R_y &= P_y Q_y \\
R_z &= P_z Q_z
\end{align*}
\]

As a simple example, we choose vectors \( \mathbf{P} \) and \( \mathbf{Q} \) to have length 1, and make them perpendicular to each other, as shown in figure k/1. If we compute the result of our new vector operation using the coordinate system shown in k/2, we find:

\[
\begin{align*}
R_x &= 0 \\
R_y &= 0 \\
R_z &= 0
\end{align*}
\]

The \( x \)-component is zero because \( P_x = 0 \), the \( y \)-component is zero because \( Q_y = 0 \), and the \( z \)-component is of course zero because both vectors are in the \( x-y \)-plane. However, if we carry out the same operations in coordinate system k/3, rotated 45 degrees with respect to the previous one, we find

\[
\begin{align*}
R_x &= -\frac{1}{2} \\
R_y &= \frac{1}{2} \\
R_z &= 0
\end{align*}
\]

The operation's result depends on what coordinate system we use, and since the two versions of \( \mathbf{R} \) have different lengths (one being zero and the other nonzero), they don't just represent the same answer expressed in two different coordinate systems. Such an operation will never be useful in physics, because experiments show physics works the same regardless of which way we orient the laboratory building! The useful vector operations, such as addition and scalar multiplication, are rotationally invariant, i.e., come out the same regardless of the orientation of the coordinate system.

All the vector techniques can be applied to any kind of vector, but the graphical representation of vectors as arrows is particularly natural for vectors that represent lengths and distances. We define a vector called \( \langle \mathbf{r} \rangle \) whose components are the coordinates of a particular point in space, \( \langle x \rangle \), \( \langle y \rangle \), and \( \langle z \rangle \). The \( \langle \Delta \mathbf{r} \rangle \) vector, whose components are \( \langle \Delta x \rangle \), \( \langle \Delta y \rangle \), and \( \langle \Delta z \rangle \), can then be used to represent motion that starts at one point and ends at another. Adding two \( \langle \Delta \mathbf{r} \rangle \) vectors is interpreted as a trip with two legs: by computing the \( \langle \Delta \mathbf{r} \rangle \) vector going from point A to point B plus the vector from B to C, we find the vector that would have taken us directly from A to C.
Calculations with magnitude and direction

If you ask someone where Las Vegas is compared to Los Angeles, she is unlikely to say that the \((\Delta x)\) is 290 km and the \((\Delta y)\) is 230 km, in a coordinate system where the positive \((x)\) axis is east and the \((y)\) axis points north. She will probably say instead that it's 370 km to the northeast. If she was being precise, she might specify the direction as \(38°\) counterclockwise from east. In two dimensions, we can always specify a vector's direction like this, using a single angle. A magnitude plus an angle suffice to specify everything about the vector. The following two examples show how we use trigonometry and the Pythagorean theorem to go back and forth between the \((x)\)-(\(y)\) and magnitude-angle descriptions of vectors.

**Example 58: Finding magnitude and angle from components**

Given that the \(\mathbf{\Delta r}\) vector from LA to Las Vegas has \((\Delta x)=290\) km and \((\Delta y)=230\) km, how would we find the magnitude and direction of \(\mathbf{\Delta r}\)?

We find the magnitude of \(\mathbf{\Delta r}\) from the Pythagorean theorem:

\[
|\mathbf{\Delta r}| = \sqrt{(\Delta x)^2 + (\Delta y)^2} = 370 \text{ km}
\]

We know all three sides of the triangle, so the angle \(\theta\) can be found using any of the inverse trig functions. For example, we know the opposite and adjacent sides, so

\[
\theta = \tan^{-1} \frac{\Delta y}{\Delta x} = 38°.
\]

**Example 59: Finding the components from the magnitude and angle**

Given that the straight-line distance from Los Angeles to Las Vegas is 370 km, and that the angle \(\theta\) in the figure is 38°, how can the \((x)\) and \((y)\) components of the \(\mathbf{\Delta r}\) vector be found?
The sine and cosine of $\theta$ relate the given information to the information we wish to find:

\[
\begin{align*}
\text{cos}\ \theta &= \frac{\Delta x}{|\Delta \mathbf{r}|} \\
\text{sin}\ \theta &= \frac{\Delta y}{|\Delta \mathbf{r}|}
\end{align*}
\]

Solving for the unknowns gives

\[
\begin{align*}
\Delta x &= |\Delta \mathbf{r}| \cos\ \theta \\
&= 290 \text{ km} \\
\Delta y &= |\Delta \mathbf{r}| \sin\ \theta \\
&= 230 \text{ km}
\end{align*}
\]

The following example shows the correct handling of the plus and minus signs, which is usually the main cause of mistakes by students.

**Example 60: Negative components**

San Diego is 120 km east and 150 km south of Los Angeles. An airplane pilot is setting course from San Diego to Los Angeles. At what angle should she set her course, measured counterclockwise from east, as shown in the figure?

If we make the traditional choice of coordinate axes, with $\Delta x$ pointing to the right and $\Delta y$ pointing up on the map, then her $\Delta x$ is negative, because her final $x$ value is less than her initial $x$ value. Her $\Delta y$ is positive, so we have

\[
\begin{align*}
\Delta x &= -120 \text{ km} \\
\Delta y &= 150 \text{ km}
\end{align*}
\]

If we work by analogy with the example 59, we get

\[
\theta = \tan^{-1} \left( \frac{\Delta y}{\Delta x} \right) = \tan^{-1}(-1.25) \approx -51^\circ
\]

According to the usual way of defining angles in trigonometry, a negative result means an angle that lies clockwise from the $x$ axis, which would have her heading for the Baja California. What went wrong? The answer is that when you ask your calculator to take the arctangent of a number, there are always two valid possibilities differing by 180°. That is, there are two possible angles whose tangents equal -1.25:

\[
\theta = \tan^{-1} \left( -1.25 \right) \\
\theta = -129^\circ
\]

You calculator doesn't know which is the correct one, so it just picks one. In this case, the one it picked was the wrong one, and it was up to you to add $(180^\circ)$ to it to find the right answer.

**Example 61: A shortcut**
A split second after nine o'clock, the hour hand on a clock dial has moved clockwise past the nine-o'clock position by some imperceptibly small angle $\phi$. Let positive $x$ be to the right and positive $y$ up. If the hand, with length $\ell$, is represented by a vector going from the dial's center to the tip of the hand, find this vector's $x$.

The following shortcut is the easiest way to work out examples like these, in which a vector's direction is known relative to one of the axes. We can tell that $\Delta r$ will have a large, negative $x$ component and a small, positive $y$. Since $\Delta x < 0$, there are really only two logical possibilities: either $\Delta x = -\ell \cos \phi$, or $\Delta x = -\ell \sin \phi$. Because $\phi$ is small, $\cos \phi$ is large and $\sin \phi$ is small. We conclude that $\Delta x = -\ell \cos \phi$.

A typical application of this technique to force vectors is given in example 71 on p. 205.

**Addition of vectors given their components**

The easiest type of vector addition is when you are in possession of the components, and want to find the components of their sum.

**Example 62: San Diego to Las Vegas**

Given the $\Delta x$ and $\Delta y$ values from the previous examples, find the $\Delta x$ and $\Delta y$ from San Diego to Las Vegas.

\[
\begin{align*}
\Delta x_{\text{total}} &= \Delta x_1 + \Delta x_2 \\
&= -120 \text{ km} + 290 \text{ km} \\
&= 170 \text{ km} \\
\Delta y_{\text{total}} &= \Delta y_1 + \Delta y_2 \\
&= 150 \text{ km} + 230 \text{ km} \\
&= 380 \text{ km}
\end{align*}
\]
Addition of vectors given their magnitudes and directions

In this case, you must first translate the magnitudes and directions into components, and then add the components.

Graphical addition of vectors

Often the easiest way to add vectors is by making a scale drawing on a piece of paper. This is known as graphical addition, as opposed to the analytic techniques discussed previously.

Example 63: From San Diego to Las Vegas, graphically

\[ \Delta \mathbf{r} \] Given the magnitudes and angles of the \( \Delta \mathbf{r} \) vectors from San Diego to Los Angeles and from Los Angeles to Las Vegas, find the magnitude and angle of the \( \Delta \mathbf{r} \) vector from San Diego to Las Vegas.
Using a protractor and a ruler, we make a careful scale drawing, as shown in figure o on page 200. A scale of 1 cm $\leftrightarrow$ 10 km was chosen for this solution. With a ruler, we measure the distance from San Diego to Las Vegas to be 3.8 cm, which corresponds to 380 km. With a protractor, we measure the angle $\theta$ to be 71°.

Even when we don’t intend to do an actual graphical calculation with a ruler and protractor, it can be convenient to diagram the addition of vectors in this way, as shown in figure p. With $\Delta \mathbf{r}$ vectors, it intuitively makes sense to lay the vectors tip-to-tail and draw the sum vector from the tail of the first vector to the tip of the second vector. We can do the same when adding other vectors such as force vectors.

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**Unit vector notation**

When we want to specify a vector by its components, it can be cumbersome to have to write the algebra symbol for each component:

$$\begin{equation*} \Delta x = 290 \text{ km}, \ \Delta y = 230 \text{ km} \end{equation*}$$

A more compact notation is to write

$$\begin{equation*} \Delta \mathbf{r} = (290 \text{ km}) \hat{x} + (230 \text{ km}) \hat{y}, \end{equation*}$$

where the vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$, called the unit vectors, are defined as the vectors that have magnitude equal to 1 and directions lying along the $x$, $y$, and $z$ axes. In speech, they are referred to as...
to as “x-hat,” “y-hat,” and “z-hat.”

A slightly different, and harder to remember, version of this notation is unfortunately more prevalent. In this version, the unit vectors are called \(\hat{i}\), \(\hat{j}\), and \(\hat{k}\):

\[
\Delta \mathbf{r} = (290 \ \text{km}) \hat{i} + (230 \ \text{km}) \hat{j}.
\]

Applications to relative motion, momentum, and force

![Vector Addition Diagram]

Adding vectors graphically by placing them tip-to-tail, like a train.

Vector addition is the correct way to generalize the one-dimensional concept of adding velocities in relative motion, as shown in the following example:

**Example 64: Velocity vectors in relative motion**

\(\triangleright\) You wish to cross a river and arrive at a dock that is directly across from you, but the river’s current will tend to carry you downstream. To compensate, you must steer the boat at an angle. Find the angle \(\theta\), given the magnitude, \(|\mathbf{v}_{WL}|\), of the water’s velocity relative to the land, and the maximum speed, \(|\mathbf{v}_{BW}|\), of which the boat is capable relative to the water.

\(\triangleright\) The boat’s velocity relative to the land equals the vector sum of its velocity with respect to the water and the water’s velocity with respect to the land,

\[
\mathbf{v}_{BL} = \mathbf{v}_{BW} + \mathbf{v}_{WL}.
\]

If the boat is to travel straight across the river, i.e., along the \((y)\) axis, then we need to have \(|\mathbf{v}_{BL,x}| = 0\). This \((x)\) component equals the sum of the \((x)\) components of the other two vectors,

\[
\mathbf{v}_{BL,x} = \mathbf{v}_{BW,x} + \mathbf{v}_{WL,x},
\]

or

\[
0 = -|\mathbf{v}_{BW}| \sin \theta + |\mathbf{v}_{WL}|.
\]

Solving for \(\theta\), we find

\[
\sin \theta = \frac{|\mathbf{v}_{WL}|}{|\mathbf{v}_{BW}|}.
\]

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Example 65: How to generalize one-dimensional equations

How can the one-dimensional relationships

\[
p_{\text{total}} = m_{\text{total}} \: v_{\text{cm}} \text{ and } x_{\text{cm}} = \frac{\sum_{j} m_{j} x_{j}}{\sum_{j} m_{j}}
\]

be generalized to three dimensions?

Momentum and velocity are vectors, since they have directions in space. Mass is a scalar. If we rewrite the first equation to show the appropriate quantities notated as vectors,

\[
\mathbf{p}_{\text{total}} = m_{\text{total}} \: \mathbf{v}_{\text{cm}},
\]

we get a valid mathematical operation, the multiplication of a vector by a scalar. Similarly, the second equation becomes

\[
\mathbf{r}_{\text{cm}} = \frac{\sum_{j} m_{j} \mathbf{r}_{j}}{\sum_{j} m_{j}}
\]

which is also valid. Each term in the sum on top contains a vector multiplied by a scalar, which gives a vector. Adding up all these vectors gives a vector, and dividing by the scalar sum on the bottom gives another vector.

This kind of wave-the-magic-wand-and-write-it-all-in-bold-face technique will always give the right generalization from
one dimension to three, provided that the result makes sense mathematically --- if you find yourself doing something nonsensical, such as adding a scalar to a vector, then you haven't found the generalization correctly.

**Example 66: Colliding coins**

Let the velocity vector of the incoming coin be \( \mathbf{a} \), and let the two outgoing velocity vectors be \( \mathbf{b} \) and \( \mathbf{c} \). Since the masses are the same, conservation of momentum amounts to \( \mathbf{a} = \mathbf{b} + \mathbf{c} \), which means that it has to be possible to assemble the three vectors into a triangle. If we assume that no energy is converted into heat and sound, then conservation of energy gives (discarding the common factor of \( (m/2) \)) \( a^2 = b^2 + c^2 \) for the magnitudes of the three vectors. This is the Pythagorean theorem, which will hold only if the three vectors form a right triangle.

The fact that we observe the angle to be somewhat less than 90 degrees shows that the assumption used in the proof is only approximately valid: a little energy is converted into heat and sound. The opposite case would be a collision between two blobs of putty, where the maximum possible amount of energy is converted into heat and sound, the two blobs fly off together, giving an angle of zero between their momentum vectors. The real-life experiment interpolates between the ideal extremes of 0 and 90 degrees, but comes much closer to 90.

Force is a vector, and we add force vectors when more than one force acts on the same object.

**Example 67: Pushing a block up a ramp**

We analyzed this simple machine in example 38 on page 168 using the concept of work. Here we'll do it using vector addition of forces. Figure r/2 shows the other two forces acting on the block: a normal force, \( (F_n) \), created by the ramp, and the gravitational force, \( (F_g) \). Because the block is being pushed up at constant speed, it has zero acceleration, and the total force on it must be zero. In figure r/3, we position all the force vectors tip-to-tail for addition. Since they have to add up to zero, they must join up without leaving a gap, so they form a triangle. Using trigonometry we find

\[
\begin{align*}
F_{\text{\(a\)}} &= F_g \sin \\theta \\
&= mg \sin \\theta .
\end{align*}
\]
Example 68: Buoyancy, again

In example 10 on page 85, we found that the energy required to raise a cube immersed in a fluid is as if the cube's mass had been reduced by an amount equal to the mass of the fluid that otherwise would have been in the volume it occupies (Archimedes' principle). From the energy perspective, this effect occurs because raising the cube allows a certain amount of fluid to move downward, and the decreased gravitational energy of the fluid tends to offset the increased gravitational energy of the cube. The proof given there, however, could not easily be extended to other shapes.

Thinking in terms of force rather than energy, it becomes easier to give a proof that works for any shape. A certain upward force is needed to support the object in figure s. If this force was applied, then the object would be in equilibrium: the vector sum of all the forces acting on it would be zero. These forces are $\mathbf{F}_a$, the upward force just mentioned, $\mathbf{F}_g$, the downward force of gravity, and $\mathbf{F}_f$, the total force from the fluid:

$$\mathbf{F}_a + \mathbf{F}_g + \mathbf{F}_f = 0$$

Since the fluid is under more pressure at a greater depth, the part of the fluid underneath the object tends to make more force than the part above, so the fluid tends to help support the object.
Now suppose the object was removed, and instantly replaced with an equal volume of fluid. The new fluid would be in equilibrium without any force applied to hold it up, so

\[
\begin{equation*}
\mathbf{F}_{gf} + \mathbf{F}_f = 0,
\end{equation*}
\]

where \(\mathbf{F}_{gf}\), the weight of the fluid, is not the same as \(\mathbf{F}_g\), the weight of the object, but \(\mathbf{F}_f\) is the same as before, since the pressure of the surrounding fluid is the same as before at any particular depth. We therefore have

\[
\begin{equation*}
\mathbf{F}_a = -\left(\mathbf{F}_g - \mathbf{F}_{gf}\right),
\end{equation*}
\]

which is Archimedes' principle in terms of force: the force required to support the object is lessened by an amount equal to the weight of the fluid that would have occupied its volume.

By the way, the word “pressure” that I threw around casually in the preceding example has a precise technical definition: force per unit area. The SI units of pressure are \(\text{N}/\text{m}^2\), which can be abbreviated as pascals, 1 Pa = 1 \(\text{N}/\text{m}^2\). Atmospheric pressure is about 100 kPa. By applying the equation \(\mathbf{F}_g + \mathbf{F}_f = 0\) to the top and bottom surfaces of a cubical volume of fluid, one can easily prove that the difference in pressure between two different depths is \(\Delta P = \rho g \Delta y\). (In physics, “fluid” can refer to either a gas or a liquid.)

Archimedes' principle works regardless of whether the object is a cube. The fluid makes a force on every square millimeter of the object's surface.

Pressure is discussed in more detail in chapter 5.

**Example 69: A solar sail**
A solar sail, figure 1/1, allows a spacecraft to get its thrust without using internal stores of energy or having to carry along mass that it can shove out the back like a rocket. Sunlight strikes the sail and bounces off, transferring momentum to the sail. A working 30-meter-diameter solar sail, Cosmos 1, was built by an American company, and was supposed to be launched into orbit aboard a Russian booster launched from a submarine, but launch attempts in 2001 and 2005 both failed.

An artist's rendering of what Cosmos 1 would have looked like in orbit.
In this example, we will calculate the optimal orientation of the sail, assuming that “optimal” means changing the vehicle's energy as rapidly as possible. For simplicity, we model the complicated shape of the sail's surface as a disk, seen edge-on in figure t/2, and we assume that the craft is in a nearly circular orbit around the sun, hence the 90-degree angle between the direction of motion and the incoming sunlight. We assume that the sail is 100% reflective. The orientation of the sail is specified using the angle $\theta$ between the incoming rays of sunlight and the perpendicular to the sail. In other words, $\theta=0$ if the sail is catching the sunlight full-on, while $\theta=90^\circ$ means that the sail is edge-on to the sun.

Conservation of momentum gives
\[
\begin{align*}
\mathbf{p}_{\text{light},i} &= \mathbf{p}_{\text{light},f} + \Delta \mathbf{p}_{\text{sail}}, \\
\text{where } \Delta \mathbf{p}_{\text{sail}} \text{ is the change in momentum picked up by the sail.} \\
\text{Breaking this down into components, we have} \\
0 &= p_{\text{light},f,x} + \Delta p_{\text{sail},x} \\
\text{and} \\
p_{\text{light},i,y} &= p_{\text{light},f,y} + \Delta p_{\text{sail},y}.
\end{align*}
\]

As in example 53 on page 189, the component of the force that is directly away from the sun (up in figure t/2) doesn't change the energy of the craft, so we only care about $\Delta p_{\text{sail},x}$, which equals $-p_{\text{light},f,x}$. The outgoing light ray forms an angle of $2\theta$ with the negative $y$ axis, or $270^\circ-2\theta$ measured counterclockwise from the $x$ axis, so the useful thrust depends on $-\cos(270^\circ-2\theta) = \sin 2\theta$.

However, this is all assuming a given amount of light strikes the sail. During a certain time period, the amount of sunlight striking the sail depends on the cross-sectional area the sail presents to the sun, which is proportional to $\cos \theta$. For $\theta=90^\circ$, $\cos \theta$ equals zero, since the sail is edge-on to the sun.

Putting together these two factors, the useful thrust is proportional to $\sin 2\theta \cos \theta$, and this quantity is maximized for $\theta \approx 35^\circ$. A counterintuitive fact about this maneuver is that as the spacecraft spirals outward, its total energy (kinetic plus gravitational) increases, but its kinetic energy actually decreases!

**Example 70: A layback**

The figure shows a rock climber using a technique called a layback. He can make the normal forces $F_{N1}$ and $F_{N2}$ large, which has the side-effect of increasing the frictional forces $F_{F1}$ and $F_{F2}$, so that he doesn't slip down due to the gravitational (weight) force $F_W$. The purpose of the problem is not to analyze all of this in detail, but simply to practice finding the components of the forces based on their magnitudes. To keep the notation simple, let's write $F_{N1}$ for $|F_{N1}|$, etc. The crack overhangs by a small, positive angle $\theta \approx 9^\circ$.

In this example, we determine the $(x)$ component of $F_{N1}$. The other nine components are left as an exercise to the reader (problem 81, p. 236).

The easiest method is the one demonstrated in example 62 on p. 199. Casting vector $F_{N1}$'s shadow on the...
ground, we can tell that it would point to the left, so its \( (x) \) component is negative. The only two possibilities for its \( (x) \) component are therefore \(-F_{N1}\cos\theta\) or \(-F_{N1}\sin\theta\). We expect this force to have a large \( (x) \) component and a much smaller \( (y) \). Since \( \cos\theta \) is small, \( \cos\theta \approx 1 \), while \( \sin\theta \) is small. Therefore the \( (x) \) component must be \(-F_{N1}\cos\theta\).

v / Example 71 and problem 81 on p. 236.

Discussion Questions

◊ An object goes from one point in space to another. After it arrives at its destination, how does the magnitude of its \( \Delta r \) vector compare with the distance it traveled?

◊ In several examples, I've dealt with vectors having negative components. Does it make sense as well to talk about negative and positive vectors?

◊ If you're doing graphical addition of vectors, does it matter which vector you start with and which vector you start from the other vector's tip?

◊ If you add a vector with magnitude 1 to a vector of magnitude 2, what magnitudes are possible for the vector sum?

◊ Which of these examples of vector addition are correct, and which are incorrect?

w / Discussion question E.
◊ Is it possible for an airplane to maintain a constant velocity vector but not a constant $|v|$? How about the opposite -- a constant $|v|$ but not a constant velocity vector? Explain.

◊ New York and Rome are at about the same latitude, so the earth's rotation carries them both around nearly the same circle. Do the two cities have the same velocity vector (relative to the center of the earth)? If not, is there any way for two cities to have the same velocity vector?

◊ The figure shows a roller coaster car rolling down and then up under the influence of gravity. Sketch the car's velocity vectors and acceleration vectors. Pick an interesting point in the motion and sketch a set of force vectors acting on the car whose vector sum could have resulted in the right acceleration vector.

![Roller Coaster](image)

$x$ / Discussion question H.

◊ The following is a question commonly asked by students:

“Why does the force vector always have to point in the same direction as the acceleration vector? What if you suddenly decide to change your force on an object, so that your force is no longer pointing in the same direction that the object is accelerating?”

What misunderstanding is demonstrated by this question? Suppose, for example, a spacecraft is blasting its rear main engines while moving forward, then suddenly begins firing its sideways maneuvering rocket as well. What does the student think Newton's laws are predicting?

◊ Debug the following incorrect solutions to this vector addition problem.

*Problem:* Freddi Fish\(^\text{TM}\) swims 5.0 km northeast, and then 12.0 km in the direction 55 degrees west of south. How far does she end up from her starting point, and in what direction is she from her starting point?

*Incorrect solution #1:*

$5.0 \text{ km} + 12.0 \text{ km} = 17.0 \text{ km}$

*Incorrect solution #2:*

$\sqrt{(5.0 \text{ km})^2 + (12.0 \text{ km})^2} = 13.0 \text{ km}$

*Incorrect solution #3:*

Let $\mathbf{A}$ and $\mathbf{B}$ be her two $\Delta \mathbf{r}$ vectors, and let
\[ \mathbf{C} = \mathbf{A} + \mathbf{B} \]. Then
\[
\begin{align*}
A_x &= (5.0 \ \text{km}) \cos 45° = 3.5 \ \text{km} \\
B_x &= (12.0 \ \text{km}) \cos 55° = 6.9 \ \text{km} \\
A_y &= (5.0 \ \text{km}) \sin 45° = 3.5 \ \text{km} \\
B_y &= (12.0 \ \text{km}) \sin 55° = 9.8 \ \text{km} \\
C_x &= A_x + B_x \\
\ &= 10.4 \ \text{km} \\
C_y &= A_y + B_y \\
\ &= 13.3 \ \text{km} \\
|\mathbf{C}| &= \sqrt{C_x^2 + C_y^2} \\
\ &= 16.9 \ \text{km} \\
\text{direction} &= \tan^{-1} \left( \frac{13.3}{10.4} \right) \approx 52° \ \text{north of east} \\
\end{align*}
\]

Incorrect solution #4:
(same notation as above)
\[
\begin{align*}
A_x &= (5.0 \ \text{km}) \cos 45° = 3.5 \ \text{km} \\
B_x &= -(12.0 \ \text{km}) \cos 55° = -6.9 \ \text{km} \\
A_y &= (5.0 \ \text{km}) \sin 45° = 3.5 \ \text{km} \\
B_y &= -(12.0 \ \text{km}) \sin 55° = -9.8 \ \text{km} \\
C_x &= A_x + B_x \\
\ &= -3.4 \ \text{km} \\
C_y &= A_y + B_y \\
\ &= -6.3 \ \text{km} \\
|\mathbf{C}| &= \sqrt{C_x^2 + C_y^2} \\
\ &= 7.2 \ \text{km} \\
\text{direction} &= \tan^{-1} \left( \frac{-6.3}{-3.4} \right) \approx 62° \ \text{north of east} \\
\end{align*}
\]

Incorrect solution #5:
(same notation as above)
\[
\begin{align*}
A_x &= (5.0 \ \text{km}) \cos 45° = 3.5 \ \text{km} \\
B_x &= -(12.0 \ \text{km}) \sin 55° = -9.8 \ \text{km} \\
A_y &= (5.0 \ \text{km}) \sin 45° = 3.5 \ \text{km} \\
B_y &= -(12.0 \ \text{km}) \cos 55° = -6.9 \ \text{km} \\
C_x &= A_x + B_x \\
\ &= -6.3 \ \text{km} \\
C_y &= A_y + B_y \\
\ &= -3.4 \ \text{km} \\
|\mathbf{C}| &= \sqrt{C_x^2 + C_y^2} \\
\ &= 7.2 \ \text{km} \\
\text{direction} &= \tan^{-1} \left( \frac{-3.4}{-6.3} \right) \approx 28° \ \text{north of east} \\
\end{align*}
\]

### 3.4.4 Calculus with vectors

#### Differentiation

In one dimension, we define the velocity as the derivative of the position with respect to time, and we can think of the derivative as what we get when we calculate \(\frac{\Delta x}{\Delta t}\) for very short time intervals. The quantity \(\Delta x = x_f - x_i\) is calculated by subtraction. In three dimensions, \(x\) becomes \(\mathbf{r}\), and the \(\Delta\mathbf{r}\) vector is calculated by vector subtraction, \(\Delta\mathbf{r} = \mathbf{r}_f - \mathbf{r}_i\). Vector subtraction is defined component by component, so when we take the derivative of a vector, this means we end up taking the derivative component by component,
\[
\begin{equation*}
v_x = \frac{dx}{dt} , v_y = \frac{dy}{dt}, v_z = \frac{dz}{dt} \\
\end{equation*}
\]
or
\[
\begin{equation*}
\frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z} . \\
\end{equation*}
\]

All of this reasoning applies equally well to any derivative of a vector, so for instance we can take the second derivative,
\[
\begin{equation*}
a_x = \frac{dv_x}{dt} , a_y = \frac{dv_y}{dt} , a_z = \frac{dv_z}{dt} \\
\end{equation*}
\]
or
\[
\begin{equation*}
\frac{d\mathbf{v}}{dt} = \frac{dv_x}{dt}\hat{x} + \frac{dv_y}{dt}\hat{y} + \frac{dv_z}{dt}\hat{z} . \\
\end{equation*}
\]
A counterintuitive consequence of this is that the acceleration vector does not need to be in the same direction as the motion. The velocity vector points in the direction of motion, but by Newton's second law, $\mathbf{a} = \frac{\mathbf{F}}{m}$, the acceleration vector points in the same direction as the force, not the motion. This is easiest to understand if we take velocity vectors from two different moments in the motion, and visualize subtracting them graphically to make a $\Delta\mathbf{v}$ vector. The direction of the $\Delta\mathbf{v}$ vector tells us the direction of the acceleration vector as well, since the derivative $\frac{d\mathbf{v}}{dt}$ can be approximated as $\frac{\Delta\mathbf{v}}{\Delta t}$. As shown in figure y/1, a change in the magnitude of the velocity vector implies an acceleration that is in the direction of motion. A change in the direction of the velocity vector produces an acceleration perpendicular to the motion, y/2.

**Example 71: Circular motion**

An object moving in a circle of radius $r$ in the $(x)$-$y$ plane has

\[
\begin{align*}
x &= r \cos \omega t \\
y &= r \sin \omega t 
\end{align*}
\]

where $\omega$ is the number of radians traveled per second, and the positive or negative sign indicates whether the motion is clockwise or counterclockwise. What is its acceleration?

The components of the velocity are

\[
\begin{align*}
v_x &= -\omega r \sin \omega t \\
v_y &= \omega r \cos \omega t 
\end{align*}
\]
and for the acceleration we have

\[
\begin{align*}
  a_x &= -\omega^2 r \cos \omega t \\
  a_y &= -\omega^2 r \sin \omega t .
\end{align*}
\]

The acceleration vector has cosines and sines in the same places as the \( r \) vector, but with minus signs in front, so it points in the opposite direction, i.e., toward the center of the circle. By Newton's second law, \( a = F/(m) \), this shows that the force must be inward as well; without this force, the object would fly off straight.

The magnitude of the acceleration is

\[
|\mathbf{a}| = \sqrt{a_x^2 + a_y^2} = \omega^2 r .
\]

It makes sense that \( \omega^2 \) is squared, since reversing the sign of \( \omega \) corresponds to reversing the direction of motion, but the acceleration is toward the center of the circle, regardless of whether the motion is clockwise or counterclockwise. This result can also be rewritten in the form

\[
|\mathbf{a}| = \frac{|\mathbf{v}|^2}{r} .
\]

Although I've relegated the results \( a = \omega^2 r = |\mathbf{v}|^2/r \) to an example because they are a straightforward corollary of more general principles already developed, they are important and useful enough to record for later use.

These results are counterintuitive as well. Until Newton, physicists and laypeople alike had assumed that the planets would
need a force to push them *forward* in their orbits. Figure \( z \) may help to make it more plausible that only an inward force is required. A forward force might be needed in order to cancel out a backward force such as friction, \( \text{aa} \), but the total force in the forward-backward direction needs to be exactly zero for constant-speed motion.

\[ \text{aa} / \text{The total force in the forward-backward direction is zero in both cases.} \]

When you are in a car undergoing circular motion, there is also a strong illusion of an *outward* force. But what object could be making such a force? The car's seat makes an inward force on you, not an outward one. There is no object that could be exerting an outward force on your body. In reality, this force is an illusion that comes from our brain's intuitive efforts to interpret the situation within a noninertial frame of reference. As shown in figure \( \text{ab} \), we can describe everything perfectly well in an inertial frame of reference, such as the frame attached to the sidewalk. In such a frame, the bowling ball goes straight because there is *no* force on it. The wall of the truck's bed hits the ball, not the other way around.
There is no outward force on the bowling ball, but in the noninertial frame it seems like one exists.

Integration

An integral is really just a sum of many infinitesimally small terms. Since vector addition is defined in terms of addition of the components, an integral of a vector quantity is found by doing integrals component by component.

Example 72: Projectile motion

\[
\begin{align*}
x &= \int\left(\int{a_x\ \text{d}t}\right)\text{d}t \\
&= \int\left(a_xt+v_{x\text{o}}\right)\text{d}t,
\end{align*}
\]

where \(v_{x\text{o}}\) is a constant of integration, and
\[
x &= \frac{1}{2}a_xt^2 + v_{x\text{o}}t + x_{\text{o}}.
\]

Similarly, \(y = (1/2) a_y t^2 + v_{y\text{o}} t + y_{\text{o}}\) and \(z = (1/2) a_z t^2 + v_{z\text{o}} t + z_{\text{o}}\). Once one has gained a little confidence, it becomes natural to do the whole thing as a single vector integral,

\[
\begin{align*}
\mathbf{r} &= \int\left(\int{\mathbf{a}\ \text{d}t}\right)\text{d}t \\
&= \int\left(\mathbf{a}t+\mathbf{v}_\text{o}\right)\text{d}t \\
&= \frac{1}{2}\mathbf{a} t^2+\mathbf{v}_\text{o} t+\mathbf{r}_\text{o}.
\end{align*}
\]
where now the constants of integration are vectors.

**Discussion Questions**

◊

In the game of crack the whip, a line of people stand holding hands, and then they start sweeping out a circle. One person is at the center, and rotates without changing location. At the opposite end is the person who is running the fastest, in a wide circle. In this game, someone always ends up losing their grip and flying off. Suppose the person on the end loses her grip. What path does she follow as she goes flying off? (Assume she is going so fast that she is really just trying to put one foot in front of the other fast enough to keep from falling; she is not able to get any significant horizontal force between her feet and the ground.)

◊ Suppose the person on the outside is still holding on, but feels that she may lose her grip at any moment. What force or forces are acting on her, and in what directions are they? (We are not interested in the vertical forces, which are the earth's gravitational force pulling down, and the ground's normal force pushing up.) Make a table in the format shown in subsection 3.2.6.

◊ Suppose the person on the outside is still holding on, but feels that she may lose her grip at any moment. What is wrong with the following analysis of the situation? “The person whose hand she's holding exerts an inward force on her, and because of Newton's third law, there's an equal and opposite force acting outward. That outward force is the one she feels throwing her outward, and the outward force is what might make her go flying off, if it's strong enough.”

◊ If the only force felt by the person on the outside is an inward force, why doesn't she go straight in?

◊
In the amusement park ride shown in the figure, the cylinder spins faster and faster until the customer can pick her feet up off the floor without falling. In the old Coney Island version of the ride, the floor actually dropped out like a trap door, showing the ocean below. (There is also a version in which the whole thing tilts up diagonally, but we're discussing the version that stays flat.) If there is no outward force acting on her, why does she stick to the wall? Analyze all the forces on her.

◊ What is an example of circular motion where the inward force is a normal force? What is an example of circular motion where the inward force is friction? What is an example of circular motion where the inward force is the sum of more than one force?

◊ Does the acceleration vector always change continuously in circular motion? The velocity vector?

◊ A certain amount of force is needed to provide the acceleration of circular motion. What if we are exerting a force perpendicular to the direction of motion in an attempt to make an object trace a circle of radius \( r \), but the force isn't as big as \( (m|v|^2/r) \)?

◊ Suppose a rotating space station is built that gives its occupants the illusion of ordinary gravity. What happens when a person in the station lets go of a ball? What happens when she throws a ball straight “up” in the air (i.e., towards the center)?

### 3.4.5 The dot product

How would we generalize the mechanical work equation \( \langle dE=F \cdot dx \rangle \) to three dimensions? Energy is a scalar, but force and distance are vectors, so it might seem at first that the kind of “magic-wand” generalization discussed on page 202 failed here, since we don't know of any way to multiply two vectors together to get a scalar. Actually, this is Nature giving us a hint that there is such a multiplication operation waiting for us to invent it, and since Nature is simple, we can be assured that this operation will work just fine in any situation where a similar generalization is required.

How should this operation be defined? Let's consider what we would get by performing this operation on various combinations of the unit vectors \( \hat{\mathbf{x}} \), \( \hat{\mathbf{y}} \), and \( \hat{\mathbf{z}} \). The conventional notation for the operation is to put a dot, \( \cdot \), between the two vectors, and the operation is therefore called the dot product. Rotational invariance requires that we handle the three coordinate axes in the same way, without giving special treatment to any of them, so we must have \( \hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} \) and \( \hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{x}} \). This is supposed to be a way of generalizing ordinary multiplication, so for consistency with the property \( (1 \times 1 = 1) \) of ordinary numbers, the result of multiplying a magnitude-one vector by itself had better be the scalar 1, so \( \hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = 1 \).
\[
\hat{x} = \hat{y} \cdot \hat{y} = \hat{z} \cdot \hat{z} = 1.
\]
Furthermore, there is no way to satisfy rotational invariance unless we define the mixed products to be zero, \(\hat{x} \cdot \hat{y} = \hat{y} \cdot \hat{z} = \hat{z} \cdot \hat{x} = 0\); for example, a 90-degree rotation of our frame of reference about the \(z\) axis reverses the sign of \(\hat{x} \cdot \hat{y}\), but rotational invariance requires that \(\hat{x} \cdot \hat{y}\) produce the same result either way, and zero is the only number that stays the same when we reverse its sign. Establishing these six products of unit vectors suffices to define the operation in general, since any two vectors that we want to multiply can be broken down into components, e.g., \((2\hat{x} + 3\hat{z}) \cdot \hat{z} = 2\hat{x} \cdot \hat{z} + 3\hat{z} \cdot \hat{z} = 0 + 3 = 3\). Thus by requiring rotational invariance and consistency with multiplication of ordinary numbers, we find that there is only one possible way to define a multiplication operation on two vectors that gives a scalar as the result.\(^{17}\) The dot product has all of the properties we normally associate with multiplication, except that there is no “dot division.”

**Example 73: Dot product in terms of components**

If we know the components of any two vectors \(b\) and \(c\), we can find their dot product:

\[
\begin{align*}
\mathbf{b} \cdot \mathbf{c} &= (b_x \hat{x} + b_y \hat{y} + b_z \hat{z}) \cdot (c_x \hat{x} + c_y \hat{y} + c_z \hat{z}) \\
&= b_x c_x + b_y c_y + b_z c_z.
\end{align*}
\]

**Example 74: Magnitude expressed with a dot product**

If we take the dot product of any vector \(b\) with itself, we find

\[
\begin{align*}
\mathbf{b} \cdot \mathbf{b} &= (b_x \hat{x} + b_y \hat{y} + b_z \hat{z}) \cdot (b_x \hat{x} + b_y \hat{y} + b_z \hat{z}) \\
&= b_x^2 + b_y^2 + b_z^2,
\end{align*}
\]
so its magnitude can be expressed as \(\sqrt{\mathbf{b} \cdot \mathbf{b}}\). We will often write \(b^2\) to mean \((\mathbf{b} \cdot \mathbf{b})\), when the context makes it clear what is intended. For example, we could express kinetic energy as \((\text{1/2}) m \mathbf{v} \cdot \mathbf{v}\), \((\text{1/2}) m \mathbf{v} \cdot \mathbf{v}\), or \((\text{1/2}) m v^2\). In the third version, nothing but context tells us that \(\mathbf{v}\) really stands for the magnitude of some vector \(\mathbf{v}\).

**Example 75: Geometric interpretation**

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In figure ae, vectors \( \mathbf{a}, \mathbf{b}, \) and \( \mathbf{c} \) represent the sides of a triangle, and \( \mathbf{a} = \mathbf{b} + \mathbf{c} \). The law of cosines gives

\[
|\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2|\mathbf{a}||\mathbf{b}| \cos \theta .
\]

Using the result of example 75, we can also write this as

\[
|\mathbf{c}|^2 = \mathbf{c} \cdot \mathbf{c} = (\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - 2\mathbf{a} \cdot \mathbf{b} .
\]

Matching up terms in these two expressions, we find

\[
\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta ,
\]

which is a geometric interpretation for the dot product.

The result of example 75 is very useful. It gives us a way to find the angle between two vectors if we know their components. It can be used to show that the dot product of any two perpendicular vectors is zero. It also leads to a nifty proof that the dot product is rotationally invariant --- up until now I've only proved that if a rotationally invariant product exists, the dot product is it --- because angles and lengths aren't affected by a rotation, so the right side of the equation is rotationally invariant, and therefore so is the left side.

I introduced the whole discussion of the dot product by way of generalizing the equation \( dE = Fdx \) to three dimensions. In terms of a dot product, we have

\[
\begin{align*}
dE &= \mathbf{F} \cdot d\mathbf{r} . \\
\text{If} \ \mathbf{F} \ \text{is a constant, integrating both sides gives} \quad \Delta E &= \mathbf{F} \cdot \Delta \mathbf{r} .
\end{align*}
\]

(If that step seemed like black magic, try writing it out in terms of components.)
Breaking trail, by Walter E. Bohl. The pack horse is not doing any work on the pack, because the pack is moving in a horizontal line at constant speed, and therefore there is no kinetic or gravitational energy being transferred into or out of it.

If the force is perpendicular to the motion, as in figure af, then the work done is zero. The pack horse is doing work within its own body, but is not doing work on the pack.

**Example 76: Pushing a lawnmower**

I push a lawnmower with a force \( \mathbf{F} = (110 \text{ N}) \hat{x} - (40 \text{ N}) \hat{y} \), and the total distance I travel is \( (100 \text{ m}) \hat{x} \). How much work do I do?

The dot product is 11000 \( \text{N} \cdot \text{m} = 11000 \text{ J} \).

A good application of the dot product is to allow us to write a simple, streamlined proof of separate conservation of the momentum components. (You can skip the proof without losing the continuity of the text.) The argument is a generalization of the one-dimensional proof on page 130, and makes the same assumption about the type of system of particles we’re dealing with. The kinetic energy of one of the particles is \( \frac{1}{2}m \mathbf{v} \cdot \mathbf{v} \), and when we transform into a different frame of reference moving with velocity \( \mathbf{u} \) relative to the original frame, the one-dimensional rule \( \mathbf{v} \rightarrow \mathbf{v} + \mathbf{u} \) turns into vector addition, \( \mathbf{v} \rightarrow \mathbf{v} + \mathbf{u} \). In the new frame of reference, the kinetic energy is \( \frac{1}{2}m (\mathbf{v} + \mathbf{u}) \cdot (\mathbf{v} + \mathbf{u}) \). For a system of \( n \) particles, we have

\[
K = \sum_{j=1}^{n} \frac{1}{2}m_j (\mathbf{v}_j + \mathbf{u}) \cdot (\mathbf{v}_j + \mathbf{u})
= \frac{1}{2} \left[ \sum_{j=1}^{n} m_j \mathbf{v}_j \cdot \mathbf{v}_j + 2 \sum_{j=1}^{n} m_j \mathbf{v}_j \cdot \mathbf{u} + \sum_{j=1}^{n} m_j \mathbf{u} \cdot \mathbf{u} \right].
\]

As in the proof on page 130, the first sum is simply the total kinetic energy in the original frame of reference, and the last sum is a constant, which has no effect on the validity of the conservation law. The middle sum can be rewritten as

\[
2 \sum_{j=1}^{n} m_j \mathbf{v}_j \cdot \mathbf{u} = 2 \mathbf{u} \cdot \sum_{j=1}^{n} m_j \mathbf{v}_j = 2 \mathbf{u} \cdot \sum_{j=1}^{n} \mathbf{p}_j,
\]

so the only way energy can be conserved for all values of \( \mathbf{u} \) is if the vector sum of the momenta is conserved as well.
3.4.6 Gradients and line integrals (optional)

This subsection introduces a little bit of vector calculus. It can be omitted without loss of continuity, but the techniques will be needed in our study of electricity and magnetism, and it may be helpful to be exposed to them in easy-to-visualize mechanical contexts before applying them to invisible electrical and magnetic phenomena.

![Diagram of an object moving through a field of force.](image)

In physics we often deal with fields of force, meaning situations where the force on an object depends on its position. For instance, figure ag could represent a map of the trade winds affecting a sailing ship, or a chart of the gravitational forces experienced by a space probe entering a double-star system. An object moving under the influence of this force will not necessarily be moving in the same direction as the force at every moment. The sailing ship can tack against the wind, due to the force from the water on the keel. The space probe, if it entered from the top of the diagram at high speed, would start to curve around to the right, but its inertia would carry it forward, and it wouldn't instantly swerve to match the direction of the gravitational force. For convenience, we've defined the gravitational field, \( g \), as the force per unit mass, but that trick only leads to a simplification because the gravitational force on an object is proportional to its mass. Since this subsection is meant to apply to any kind of force, we'll discuss everything in terms of the actual force vector, \( F \), in units of newtons.

If an object moves through the field of force along some curved path from point \( \mathbf{r}_1 \) to point \( \mathbf{r}_2 \), the force will do a certain amount of work on it. To calculate this work, we can break the path up into infinitesimally short segments, find the work done along each segment, and add them all up. For an object traveling along a nice straight \( (x) \) axis, we use the symbol \( \Delta x \) to indicate the length of any infinitesimally short segment. In three dimensions, moving along a curve, each segment is a tiny vector \( \Delta \mathbf{r} = \Delta x \hat{\mathbf{x}} + \Delta y \hat{\mathbf{y}} + \Delta z \hat{\mathbf{z}} \). The work theorem can be expressed as a dot product, so the work done along a segment is \( \mathbf{F} \cdot \Delta \mathbf{r} \). We want to integrate this, but we don't know how to integrate with respect to a variable that's a vector, so let's define a variable \( s \) that indicates the distance traveled so far along the curve, and integrate with respect to it instead. The expression \( \mathbf{F} \cdot \Delta \mathbf{r} \) can be rewritten as \( |\mathbf{F}| |\Delta \mathbf{r}| \cos \theta \), where \( \theta \) is the angle between \( \mathbf{F} \) and \( \Delta \mathbf{r} \). But \( |\Delta \mathbf{r}| \) is simply \( \Delta s \), so the amount of work done becomes

\[
\Delta E = \int_{\mathbf{r}_1}^{\mathbf{r}_2} |\mathbf{F}| \cos \theta \, ds.
\]

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Both \( \mathbf{F} \) and \( \theta \) are functions of \( s \). As a matter of notation, it's cumbersome to have to write the integral like this. Vector notation was designed to eliminate this kind of drudgery. We therefore define the line integral
\[
\int_C {\mathbf{F} \cdot d\mathbf{r}}
\]
as a way of notating this type of integral. The 'C' refers to the curve along which the object travels. If we don't know this curve then we typically can't evaluate the line integral just by knowing the initial and final positions \( \mathbf{r}_1 \) and \( \mathbf{r}_2 \).

The basic idea of calculus is that integration undoes differentiation, and vice-versa. In one dimension, we could describe an interaction either in terms of a force or in terms of an interaction energy. We could integrate force with respect to position to find minus the energy, or we could find the force by taking minus the derivative of the energy. In the line integral, position is represented by a vector. What would it mean to take a derivative with respect to a vector? The correct way to generalize the derivative \( \frac{dU}{dx} \) to three dimensions is to replace it with the following vector,
\[
\nabla U = \frac{\partial U}{\partial x}\hat{x} + \frac{\partial U}{\partial y}\hat{y} + \frac{\partial U}{\partial z}\hat{z}
\]
called the gradient of \( U \), and written with an upside-down delta like this, \( \nabla U \). Each of these three derivatives is really what's known as a partial derivative. What that means is that when you're differentiating \( U \) with respect to \( x \), you're supposed to treat \( y \) and \( z \) and constants, and similarly when you do the other two derivatives. To emphasize that a derivative is a partial derivative, it's customary to write it using the symbol \( \partial \) in place of the differential d's. Putting all this notation together, we have
\[
\nabla U = \frac{\partial U}{\partial x}\hat{x} + \frac{\partial U}{\partial y}\hat{y} + \frac{\partial U}{\partial z}\hat{z}
\]
The gradient looks scary, but it has a very simple physical interpretation. It's a vector that points in the direction in which \( U \) is increasing most rapidly, and it tells you how rapidly \( U \) is increasing in that direction. For instance, sperm cells in plants and animals find the egg cells by traveling in the direction of the gradient of the concentration of certain hormones. When they reach the location of the strongest hormone concentration, they find their destiny. In terms of the gradient, the force corresponding to a given interaction energy is \( \mathbf{F} = -\nabla U \).

**Example 77: Force exerted by a spring**

In one dimension, Hooke's law is \( U = \frac{1}{2} kx^2 \). Suppose we tether one end of a spring to a post, but it's free to stretch and swing around in a plane. Let's say its equilibrium length is zero, and let's choose the origin of our coordinate system to be at the post. Rotational invariance requires that its energy only depend on the magnitude of the \( \mathbf{r} \) vector, not its direction, so in two dimensions we have \( U = \frac{1}{2} k|\mathbf{r}|^2 \). The force exerted by the spring is then
\[
\mathbf{F} = -\nabla U = -k\hat{x}|\mathbf{r}| - k\hat{y}|\mathbf{r}| = -k\hat{x}|\mathbf{r}|(x\hat{x} + y\hat{y})
\]
The magnitude of this force vector is \( k|\mathbf{r}| \), and its direction is toward the origin.
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