4.3: Linearity and Superposition

An important aspect of linear systems is that the solutions obey the Principle of Superposition, that is, for the superposition of different oscillatory modes, the amplitudes add linearly. The linearly-damped linear oscillator is an example of a linear system in that it involves only linear operators, that is, it can be written in the operator form (appendix F.2)

\[ \frac{d^2}{dt^2} + \Gamma \frac{d}{dt} + \omega_0^2 x(t) = A \cos \omega t \]

The quantity in the brackets on the left hand side is a linear operator that can be designated by \( L \) where

\[ \mathbb{L} x(t) = F(t) \]

An important feature of linear operators is that they obey the principle of superposition. This property results from the fact that linear operators are distributive, that is

\[ \mathbb{L}(x_1 + x_2) = \mathbb{L}(x_1) + \mathbb{L}(x_2) \]

Therefore if there are two solutions \( x_1(t) \) and \( x_2(t) \) for two different forcing functions \( F_1(t) \) and \( F_2(t) \)

\[ \begin{align*} \mathbb{L}x_1(t) & = & F_1(t) \\ \mathbb{L}x_1(t) & = & F_2(t) \end{align*} \]

then the addition of these two solutions, with arbitrary constants, also is a solution for linear operators.

\[ \mathbb{L}(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 F_1(t) + \alpha_2 F_2(t) \]

In general then
The left hand bracket can be identified as the linear combination of solutions

\[ x(t) = \sum_{n=1}^{N} \alpha x_n(t) \]

while the driving force is a linear superposition of harmonic forces

\[ F(t) = \sum_{n=1}^{N} \alpha_n F_n(t) \]

Thus these linear combinations also satisfy the general linear equation

\[ L x(t) = F(t) \]

Applicability of the Principle of Superposition to a system provides a tremendous advantage for handling and solving the equations of motion of oscillatory systems.