9.4: Hamiltonian in Different Coordinate Systems

Hamiltonian in different coordinate systems

Prior to solving problems using Hamiltonian mechanics, it is useful to express the Hamiltonian in cylindrical and spherical coordinates for the special case of conservative forces since these are encountered frequently in physics.

Cylindrical coordinates ($\rho, z, \phi$)

Consider cylindrical coordinates ($\rho, z, \phi$). Expressed in cartesian coordinates:
\[
\begin{align*}
  x &= \rho \cos \phi \\
  y &= \rho \sin \phi \\
  z &= z
\end{align*}
\]

Using appendix table (C.3), the Lagrangian can be written in cylindrical coordinates as
\[
L = T - U = \frac{m}{2} \left( \dot{\rho}^2 + \rho^2 \dot{\phi}^2 + \dot{z}^2 \right) - U(\rho, z, \phi)
\]

The conjugate momenta are:
\[
\begin{align*}
  p_{\rho} &= \frac{\partial L}{\partial \dot{\rho}} = m \dot{\rho} \\
  p_{\phi} &= \frac{\partial L}{\partial \dot{\phi}} = m \rho^2 \dot{\phi} \\
  p_{z} &= \frac{\partial L}{\partial \dot{z}} = m \dot{z}
\end{align*}
\]

Assume a conservative force, then $H$ is conserved. Since the transformation from cartesian to non-rotating generalized cylindrical coordinates is time independent, then $H = E$. Then using (8.32-8.35) gives the Hamiltonian in cylindrical coordinates to be:
\[
H(\mathbf{q}, \mathbf{p}, t) = \sum_{i} p_{i} \dot{q}_{i} - L(\mathbf{q}, \mathbf{\dot{q}}, t) = \left( p_{\rho} \dot{\rho} + p_{\phi} \dot{\phi} + p_{z} \dot{z} \right) - \frac{m}{2} \left( \ddot{\rho}^2 + \rho^2 \ddot{\phi}^2 + \dot{z}^2 \right) + U(\rho, z, \phi)
\]

The canonical equations of motion in cylindrical coordinates can be written as:
\[
\begin{align*}
  \dot{p}_{\rho} &= -\frac{\partial H}{\partial \rho} = \frac{p_{\phi}^2}{m \rho^3} - \frac{\partial U}{\partial \rho} \\
  \dot{p}_{\phi} &= -\frac{\partial H}{\partial \phi} = \frac{p_{\rho} p_{\phi}}{\rho^2} + \frac{\partial U}{\partial \phi} \\
  \dot{p}_{z} &= -\frac{\partial H}{\partial z} = m \ddot{z}
\end{align*}
\]
\[
\begin{align*}
\frac{\partial H}{\partial \phi} &= -\frac{\partial U}{\partial \phi} \\
\dot{p}_z &= -\frac{\partial H}{\partial z} = -\frac{\partial U}{\partial z} \\
\dot{\rho} &= \frac{\partial H}{\partial p_\rho} = \frac{p_\rho}{m} \\
\dot{\phi} &= \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{m \rho^2} \\
\dot{z} &= \frac{\partial H}{\partial p_z} = \frac{p_z}{m}
\end{align*}
\]

Note that if \(\phi\) is cyclic, that is \(\frac{\partial U}{\partial \phi} = 0\), then the angular momentum about the \(z\) axis, \((p_\phi)\), is a constant of motion. Similarly, if \(z\) is cyclic, then \((p_z)\) is a constant of motion.

**Spherical coordinates, \((r, \theta, \phi)\)**

Appendix table \((C.4)\) shows that the spherical coordinates are related to the cartesian coordinates by

\[
\begin{align*}
x &= r \sin \theta \cos \phi \\
y &= r \sin \theta \sin \phi \\
z &= r \cos \theta
\end{align*}
\]

The Lagrangian is

\[
L = T - U = \frac{m}{2} \left( \dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right) - U(r, \theta, \phi)
\]

The conjugate momenta are

\[
\begin{align*}
p_r &= \frac{\partial L}{\partial \dot{r}} = m \dot{r} \\
p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = m r^2 \dot{\theta} \\
p_\phi &= \frac{\partial L}{\partial \dot{\phi}} = m r^2 \sin^2 \theta \dot{\phi}
\end{align*}
\]

Assuming a conservative force then \(H\) is conserved. Since the transformation from cartesian to generalized spherical coordinates is time independent, then \(H = E\). Thus using \(((8.46-8.48))\) the Hamiltonian is given in spherical coordinates by

\[
\begin{align*}
H &= \frac{1}{2m} \left( p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2 \theta} \right) + U(r, \theta, \phi)
\end{align*}
\]

Then the canonical equations of motion in spherical coordinates are

\[
\begin{align*}
\dot{p}_r &= -\frac{\partial H}{\partial r} = \frac{1}{mr^3} \left( p_\theta^2 + \frac{p_\phi^2}{\sin^2 \theta} \right) - \frac{\partial U}{\partial r} \\
\dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = \frac{1}{mr^2} \left( \frac{p_\phi^2 \cos \theta}{\sin^3 \theta} \right) - \frac{\partial U}{\partial \theta} \\
\dot{p}_\phi &= -\frac{\partial H}{\partial \phi} = -\frac{\partial U}{\partial \phi} \\
\dot{r} &= \frac{\partial H}{\partial p_r} = \frac{p_r}{m} \\
\dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m r^2} \\
\dot{\phi} &= \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{m r^2 \sin^2 \theta}
\end{align*}
\]

Note that if the coordinate \(\phi\) is cyclic, that is \(\frac{\partial U}{\partial \phi} = 0\) then the angular momentum \((p_\phi)\) is conserved. Also if the \((\theta)\) coordinate is cyclic, and \((p_\theta) = 0\), that is, there is no change in the angular momentum perpendicular to the \((z)\) axis, then \((p_\phi)\) is conserved.

An especially important spherically-symmetric Hamiltonian is that for a central field. Central fields, such as the gravitational or Coulomb fields of a uniform spherical mass, or charge, distributions, are spherically symmetric and then both \((\theta)\) and \((\phi)\) are cyclic. Thus the projection of the angular momentum \((p_\phi)\) about the \((z)\) axis is conserved for these spherically symmetric potentials. In addition, since both \((p_\theta)\) and \((p_\phi)\) are conserved, then the total angular
momentum also must be conserved as is predicted by Noether’s theorem.