13.4: Inertia Tensor

The square bracket term in $((13.3.9))$ is called the **moment of inertia tensor**, $\mathbf{I}$, which is usually referred to as the **inertia tensor**

\[
I_{ij} \equiv \sum_{\alpha}^{N} m_{\alpha} \left[ \delta_{ij} \left( \sum^3_k x^2_{\alpha, k} \right) - x_{\alpha, i} x_{\alpha, j} \right]
\] \label{13.12}

In most cases it is more useful to express the components of the inertia tensor in an integral form over the mass distribution rather than a summation for \((N)\) discrete bodies. That is,

\[
I_{ij} = \int \rho (\mathbf{r}) \left( \delta_{ij} \left( \sum^3_k x^2_{k} \right) - x_i x_j \right) dV
\]

The inertia tensor is easier to understand when written in cartesian coordinates $\mathbf{r}_{\alpha} = (x_{\alpha}, y_{\alpha}, z_{\alpha})$ rather than in the form $\mathbf{r}_{\alpha} = (x_{\alpha,1}, x_{\alpha,2}, x_{\alpha,3})$. Then, the diagonal **moments of inertia** of the inertia tensor are

\[
\begin{align}
I_{xx} & \equiv \sum_{\alpha}^{N} m_{\alpha} \left[ x_{\alpha}^2 + y_{\alpha}^2 + z_{\alpha}^2 - x_{\alpha}^2 \right] = \sum_{\alpha}^{N} m_{\alpha} \left[ y_{\alpha}^2 + z_{\alpha}^2 \right] \\
I_{yy} & \equiv \sum_{\alpha}^{N} m_{\alpha} \left[ x_{\alpha}^2 + y_{\alpha}^2 + z_{\alpha}^2 - y_{\alpha}^2 \right] = \sum_{\alpha}^{N} m_{\alpha} \left[ x_{\alpha}^2 + z_{\alpha}^2 \right] \\
I_{zz} & \equiv \sum_{\alpha}^{N} m_{\alpha} \left[ x_{\alpha}^2 + y_{\alpha}^2 + z_{\alpha}^2 - z_{\alpha}^2 \right] = \sum_{\alpha}^{N} m_{\alpha} \left[ x_{\alpha}^2 + y_{\alpha}^2 \right]
\end{align}
\]
while the off-diagonal products of inertia are

\[
\begin{align}
I_{yx} & = I_{xy} \equiv - \sum^N_{\alpha} m_{\alpha} [x_{\alpha} y_{\alpha}] \\
I_{zx} & = I_{xz} \equiv - \sum^N_{\alpha} m_{\alpha} [x_{\alpha} z_{\alpha}] \\
I_{zy} & = I_{yz} \equiv - \sum^N_{\alpha} m_{\alpha} [y_{\alpha} z_{\alpha}]
\end{align}
\]

Note that the products of inertia are symmetric in that

\[I_{ij} = I_{ji}\]

The above notation for the inertia tensor allows the angular momentum \ref{13.12} to be written as

\[
L_i = \sum^3_j I_{ij} \omega_j
\]

Expanded in cartesian coordinates

\[
\begin{align}
L_x & = I_{xx} \omega_x + I_{xy} \omega_y + I_{xz} \omega_z \\
L_y & = I_{yx} \omega_x + I_{yy} \omega_y + I_{yz} \omega_z \\
L_z & = I_{zx} \omega_x + I_{zy} \omega_y + I_{zz} \omega_z
\end{align}
\]

Note that every fixed point in a body has a specific inertia tensor. The components of the inertia tensor at a specified point depend on the orientation of the coordinate frame whose origin is located at the specified fixed point. For example, the inertia tensor for a cube is very different when the fixed point is at the center of mass compared with when the fixed point is at a corner of the cube.