10.1: Introduction to Hamilton's Principle of Least Action

Introduction

Hamilton’s principle of stationary action was introduced in two papers published by Hamilton in 1834 and 1835. Hamilton’s Action Principle provides the foundation for building Lagrangian mechanics that had been pioneered 46 years earlier. Hamilton’s Principle now underlies theoretical physics and many other disciplines in mathematics and economics. In 1834 Hamilton was seeking a theory of optics when he developed both his principle of stationary action, plus the field of Hamiltonian mechanics.

Hamilton’s Action Principle is based on defining the action functional \( S \) for \( n \) generalized coordinates which are expressed by the vector \( \mathbf{q} \) and their corresponding velocity vector \( \mathbf{\dot{q}} \).

\[
S = \int_{t_{i}}^{t_{f}} L(\mathbf{q}, \mathbf{\dot{q}}, t) dt
\]

The scalar action \( S \) is a functional of the Lagrangian \( L(\mathbf{q}, \mathbf{\dot{q}}, t) \), integrated between an initial time \( t_{i} \) and final time \( t_{f} \). In principle, higher order time derivatives of the generalized coordinates could be included, but most systems in classical mechanics are described adequately by including only the generalized coordinates, plus their velocities. The definition of the action functional allows for more general Lagrangians than the standard Lagrangian \( L(\mathbf{q}, \mathbf{\dot{q}}, t) = T(\mathbf{\dot{q}}, t) - U(\mathbf{q}, t) \) that has been used throughout chapters \( 5-8 \).

Hamilton stated that the actual trajectory of a mechanical system is that given by requiring that the action functional is stationary with respect to change of the variables. The action functional is stationary when the variational principle can be written in terms of a virtual infinitessimal displacement, \( \delta \) to be

\[
\delta S = \delta \int_{t_{i}}^{t_{f}} L(\mathbf{q}, \mathbf{\dot{q}}, t) dt = 0
\]

Typically the stationary point corresponds to a minimum of the action functional. Applying variational calculus to the action functional leads to the same Lagrange equations of motion for systems as the equations derived using d’Alembert’s Principle, if the additional generalized force terms, \( \sum_{k=1}^{m} \lambda_{k} \frac{\partial g_{k}}{\partial q_{j}}(\mathbf{q}, t) + Q_{j}^{EXC} \), are
omitted in the corresponding equations of motion.

Hamilton’s Principle is a fundamental postulate of classical mechanics that is analogous to Newton’s postulated three laws of motion for Newtonian mechanics. Lagrangian mechanics was developed based on the standard Lagrangian \( L=T-U, \) and provides a remarkably powerful and consistent approach to solving the equations of motion in classical mechanics. Hamilton’s Principle extends Lagrangian mechanics to include use of more general and non-standard Lagrangians. This chapter introduces Hamilton’s Principle, plus an extension to make it time asymmetric which allows using only initial boundary conditions. Hamilton’s Principle will be used to derive both Lagrangian and Hamiltonian mechanics from the action functional as well as for the discussion of non-standard Lagrangians. This chapter concludes with a discussion of the use of Hamilton’s action principle as the foundation for deriving the equations of motion for systems, and for applications of Lagrangian and Hamiltonian mechanics in science and engineering.