13.3: Standard Lagrangian

Lagrangian mechanics, as introduced in chapters 5? 6? was based on the concepts of kinetic energy and potential energy. The derivation of Lagrangian mechanics, given in chapter 6? was based on d’Alembert’s principle of virtual work which led to the definition of the standard Lagrangian. The standard Lagrangian was defined in chapter 6?2 to be the difference between the kinetic and potential energies.

\[ (q\dot{} - q) = (q\dot{} - q) - (q - q) \] (13.29)

Hamilton extended Lagrangian mechanics by defining Hamilton’s Principle, equation 13?2, which states that a dynamical system follows a path for which the action functional is stationary, that is, time integral of the Lagrangian. Chapter 6 showed that using the standard Lagrangian in the action functional leads to the Euler-Lagrange variational equations

\[
\frac{1}{2} \mu \dot{\varphi} \dot{\varphi} + \frac{1}{4}
\]

\[
\dot{\varphi} = \Gamma + X
\]

\[
\Gamma (q, \dot{q}) \] (13.30)
The Lagrange multiplier terms handle the holonomic constraint forces and \( \mathbf{f} \) handles the remaining excluded generalized forces. Chapters 6 – 12 showed that the use of the standard Lagrangian, with the Euler-Lagrange equations (13?3)? provides a remarkably powerful and flexible way to derive second-order equations of motion for dynamical systems in classical mechanics.

Note that the Euler-Lagrange equations, expressed solely in terms of the standard Lagrangian (13?29)? that is, excluding the \( \mathbf{f} \) + \( \mathbf{P} \) terms, are valid only under the following conditions:

1. The forces acting on the system, apart from any forces of constraint, must be derivable from scalar potentials.
2. The equations of constraint must be relations that connect the coordinates of the particles and may be functions of time, that is, the constraints are holonomic.

The \( \mathbf{f} \) + \( \mathbf{P} \) terms extend the range of validity of using the standard Lagrangian in the

Lagrange-Euler equations by introducing constraint and additional force explicitly.

Chapters 6–12 exploited Lagrangian mechanics based on use of the standard definition of the Lagrangian. This chapter shows that the powerful Lagrangian formulation, using the standard Lagrangian, can be extended to include alternative non-standard Lagrangians that may be applied to dynamical systems where use of the standard definition is inapplicable. If these non-standard Lagrangians satisfy Hamilton’s Action Principle, 13?2, then they can be used with the Euler-Lagrange equations to generate the correct equations of motion, even though the Lagrangian may have no direct relation to the kinetic and potential energies as is the case for the standard Lagrangian. Currently, the development and exploitation of non-standard Lagrangians is an active field of Lagrangian mechanics.