13.4: Gauge Invariance of the Lagrangian

Note that the standard Lagrangian is not unique in that there is a continuous spectrum of equivalent standard Lagrangians that all lead to identical equations of motion. This is because the Lagrangian \( \mathcal{L} \) is a scalar quantity that is invariant with respect to coordinate transformations. The following transformations change the standard Lagrangian, but leave the equations of motion unchanged.

1. The Lagrangian is indefinite with respect to addition of a constant to the scalar potential which cancels out when the derivatives in the Euler-Lagrange differential equations are applied.

2. The Lagrangian is indefinite with respect to addition of a constant kinetic energy.

3. The Lagrangian is indefinite with respect to addition of a total time derivative of the form \( \frac{\partial^2}{\partial t^2} \rightarrow \frac{\partial^2}{\partial t^2} + \Lambda(q, \dot{q}) \) for any differentiable function \( \Lambda(q, \dot{q}) \) of the generalized coordinates plus time, that has continuous second derivatives.

This last statement can be proved by considering a transformation between two related standard Lagrangians of the form

\[
\mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L} + \Lambda(q, \dot{q})
\]

\[
\mu \rightarrow \mu' = \mu + \frac{\partial \Lambda}{\partial \dot{q}}
\]

\[
\Lambda(q, \dot{q}) \parallel
\]

\[
\frac{\partial^2}{\partial t^2} \rightarrow \frac{\partial^2}{\partial t^2} + \Lambda(q, \dot{q})
\]
This leads to a standard Lagrangian $\mathcal{L}^2$ that has the same equations of motion as $\mathcal{L}^1$ as is shown by substituting equation 13.31 into the Euler-Lagrange equations. That is,

$$\mathcal{L}^2 = \mathcal{L}^1 + \mathcal{L}(q, \dot{q}) + \mathcal{L}(q, \dot{q}) - \mathcal{L}^1$$

(13.32)
Thus even though the $\phi_1$ and $\phi_2$ are different, they are completely equivalent in that they generate identical equations of motion.

There is an unlimited range of equivalent standard Lagrangians that all lead to the same equations of motion and satisfy the requirements of the Lagrangian. That is, there is no unique choice among the wide range of equivalent standard Lagrangians expressed in terms of generalized coordinates. This discussion is an example of gauge invariance in physics. Modern theories in physics describe reality in terms of potential fields. Gauge invariance, which also is called gauge symmetry, is a property of field theory for which different underlying fields lead to identical observable quantities. Well-known examples are the static electric potential field and the gravitational potential field where any arbitrary constant can be added to these scalar potentials with zero impact on the observed static electric field or the observed gravitational field. Gauge theories constrain the laws of physics in that the impact of gauge transformations must cancel out when expressed in terms of the observables. Gauge symmetry plays a crucial role in both classical and quantal manifestations of field theory, e.g. it is the basis of the Standard Model of electroweak and strong interactions.

Equivalent Lagrangians are a clear manifestation of gauge invariance as illustrated by equations 13?31 and 13?32 which show that adding any total time derivative of a scalar function $\Lambda(q?)$ to the Lagrangian has no observable consequences on the equations of motion. That is, although addition of the total time derivative of the scalar function $\Lambda(q? ?)$ changes the value of the Lagrangian, it does not change the equations of motion for the observables derived using equivalent standard Lagrangians. In Lagrangian formulations of classical mechanics, the gauge invariance is readily apparent by direct inspection of the Lagrangian.

Example \(\PageIndex{1}\): Gauge invariance in electromagnetism

The scalar electric potential $\Phi$ and the vector potential $A$ fields in electromagnetism are examples of gauge-invariant fields. These electromagnetic-potential fields are not directly observable, that is, the electromagnetic observable quantities are the electric field $E$ and magnetic field $B$ which can be derived from the scalar and vector potential fields $\Phi$ and $A$. An advantage of using the potential fields is that they reduce the problem from 6 components, 3 each for $E$ and $A$, to 4 components, one for the scalar field $\Phi$ and 3 for the vector potential $A$. The Lagrangian for the velocity-dependent Lorentz force, given by equation 6?6? provides an example of gauge invariance. Equations 6?63 and 6?65 showed that the electric and magnetic fields can be expressed in terms of scalar and vector potentials $\Phi$ and $A$ by the relations

\[
B = \times A
\]

\[
A
\]

\[
E = -\Phi, A
\]

The equations of motion for a charge $q$ in an electromagnetic field can be obtained by using the Lagrangian
\[ ? = 2 ?v \cdot v - \Phi - A \cdot v \]

Consider the transformations \((A?\Phi) \to (A0? \Phi0)\) in the transformed Lagrangian \(?0\) where
\[ A0 = A + ?\Lambda(r??) \]
\[ \Lambda(r??) \]
\[ \Phi0 = \Phi - ?? \]

The transformed Lorentz-force Lagrangian \(?0\) is related it to the original Lorentz-force Lagrangian \(?\) by
\[ ?0 = ? + ? \]

Note that the additive term \(?? \Lambda(r??)\) is an exact time differential. Thus the Lagrangian \(?0\) is gauge invariant implying identical equations of motion are obtained using either of these equivalent Lagrangians.

The force fields \(E\) and \(B\) can be used to show that the above transformation is gauge-invariant. That is,
\[ E0 = -?\Phi0 - \]
\[ ?A0 \cdot A = -?\Phi - ?? = E \]
\[ B0 = ? \times A0 = ? \times A = B \]

That is, the additive terms due to the scalar field \(\Lambda(r??)\) cancel. Thus the electromagnetic force fields following a gauge-invariant transformation are shown to be identical in agreement with what is inferred directly by inspection of the Lagrangian.