11.5 Induced Electric Fields

Nature is simple, but the simplicity may not become evident until a hundred years after the discovery of some new piece of physics. We've already seen, on page 598, that the time-varying magnetic field in an inductor causes an electric field. This electric field is not created by charges. That argument, however, only seems clear with hindsight. The discovery of this phenomenon of induced electric fields --- fields that are not due to charges --- was a purely experimental accomplishment by Michael Faraday (1791-1867), the son of a blacksmith who had to struggle against the rigid class structure of 19th century England.

Faraday, working in 1831, had only a vague and general idea that electricity and magnetism were related to each other, based on Oersted's demonstration, a decade before, that magnetic fields were caused by electric currents.
Figure b is a simplified drawing of the following experiment, as described in Faraday's original paper: “Two hundred and three feet of copper wire ... were passed round a large block of wood; [another] two hundred and three feet of similar wire were interposed as a spiral between the turns of the first, and metallic contact everywhere prevented by twine [insulation]. One of these [coils] was connected with a galvanometer [voltmeter], and the other with a battery... When the contact was made, there was a sudden and very slight effect at the galvanometer, and there was also a similar slight effect when the contact with the battery was broken. But whilst the ... current was continuing to pass through the one [coil], no ... effect ... upon the other [coil] could be perceived, although the active power of the battery was proved to be great, by its heating the whole of its own
coil [through ordinary resistive heating] ..."

From Faraday's notes and publications, it appears that the situation in figure b/3 was a surprise to him, and he probably thought it would be a surprise to his readers, as well. That's why he offered evidence that the current was still flowing: to show that the battery hadn't just died. The induction effect occurred during the short time it took for the black coil's magnetic field to be established, b/2. Even more counterintuitively, we get an effect, equally strong but in the opposite direction, when the circuit is broken, b/4. The effect occurs only when the magnetic field is changing, and it appears to be proportional to the derivative \(\partial\mathbf{B}/\partial\mathbf{t}\), which is in one direction when the field is being established, and in the opposite direction when it collapses.

The effect is proportional to \(\partial\mathbf{B}/\partial\mathbf{t}\), but what is the effect? A voltmeter is nothing more than a resistor with an attachment for measuring the current through it. A current will not flow through a resistor unless there is some electric field pushing the electrons, so we conclude that the changing magnetic field has produced an electric field in the surrounding space. Since the white wire is not a perfect conductor, there must be electric fields in it as well. The remarkable thing about the circuit formed by the white wire is that as the electrons travel around and around, they are always being pushed forward by electric fields. This violates the loop rule, which says that when an electron makes a round trip, there is supposed to be just as much “uphill” (moving against the electric field) as “downhill” (moving with it). That's OK. The loop rule is only true for statics. Faraday's experiments show that an electron really can go around and around, and always be going “downhill,” as in the famous drawing by M.C. Escher shown in figure c. That's just what happens when you have a curly field.

When a field is curly, we can measure its curliness using a circulation. Unlike the magnetic circulation \(\Gamma_B\), the electric circulation \(\Gamma_E\) is something we can measure directly using ordinary tools. A circulation is defined by breaking up a loop into tiny segments, \(d\mathbf{s}\), and adding up the dot products of these distance vectors with the field. But when we multiply electric field by distance, what we get is an indication of the amount of work per unit charge done on a test charge that has been moved through that distance. The work per unit charge has units of volts, and it can be measured using a voltmeter, as shown in figure e, where \(\Gamma_E\) equals the sum of the voltmeter readings. Since the electric circulation is directly measurable, most people who work with circuits are more familiar with it than they are with the magnetic circulation. They usually refer to \(\Gamma_E\) using the synonym “emf,” which stands for “electromotive force,” and notate it as \(\mathcal{E}\). (This is an unfortunate piece of terminology, because its units are really volts, not newtons.) The term emf can also be used when the path is not a closed loop.
Faraday's experiment demonstrates a new relationship
\[
\Gamma_E \propto -\frac{\partial B}{\partial t},
\]
where the negative sign is a way of showing the observed left-handed relationship, \( d \).

\[ \Gamma_B \propto I_{\text{through}} \]

The relationship between the change in the magnetic field, and the electric field it produces.

This is similar to the structure of of Ampère's law:
\[
\Gamma_B \propto I_{\text{through}}
\]

which also relates the curliness of a field to something that is going on nearby (a current, in this case).

It's important to note that even though the emf, \( \Gamma_E \), has units of volts, it isn't a voltage. A voltage is a measure of the electrical energy a charge has when it is at a certain point in space. The curly nature of nonstatic fields means that this whole concept becomes nonsense. In a curly field, suppose one electron stays at home while its friend goes for a drive around the block. When they are reunited, the one that went around the block has picked up some kinetic energy, while the one who stayed at home hasn't. We simply can't define an electrical energy \( U_e = qV \) so that \( U_e + K \) stays the same for each electron. No voltage pattern, \( V \), can do this, because then it would predict the same kinetic energies for the two electrons, which is incorrect. When we're dealing with nonstatic fields, we need to think of the electrical energy in terms of the energy density of the fields themselves.
It might sound as though an electron could get a free lunch by circling around and around in a curly electric field, resulting in a violation of conservation of energy. The following examples, in addition to their practical interest, both show that energy is in fact conserved.

**Example 18: The generator**

A basic generator, $f$, consists of a permanent magnet that rotates within a coil of wire. The magnet is turned by a motor or crank, (not shown). As it spins, the magnetic field changes. This changing magnetic field results in an electric field, which has a curly pattern. This electric field pattern creates a current that whips around the coils of wire, and we can tap this current to light the lightbulb.

If the magnet was on a frictionless bearing, could we light the bulb for free indefinitely, thus violating conservation of energy? No. Mechanical work has to be done to crank the magnet, and that's where the energy comes from. If we break the light-bulb circuit, it suddenly gets easier to crank the magnet! This is because the current in the coil sets up its own magnetic field, and that field exerts a torque on the magnet. If we stopped cranking, this torque would quickly make the magnet stop turning.

**self-check:**

When you're driving your car, the engine recharges the battery continuously using a device called an alternator, which is really just a generator. Why can't you use the alternator to start the engine if your car's battery is dead?

(answer in the back of the PDF version of the book)

**Example 19: The transformer**
In example 18 on page 540, we discussed the advantages of transmitting power over electrical lines using high voltages and low currents. However, we don't want our wall sockets to operate at 10000 volts! For this reason, the electric company uses a device called a transformer, \( g \), to convert everything to lower voltages and higher currents inside your house. The coil on the input side creates a magnetic field. Transformers work with alternating current, so the magnetic field surrounding the input coil is always changing. This induces an electric field, which drives a current around the output coil.

Since the electric field is curly, an electron can keep gaining more and more energy by circling through it again and again. Thus the output voltage can be controlled by changing the number of coils of wire on the output side. Changing the number of coils on the input side also has an effect (homework problem 33).

In any case, conservation of energy guarantees that the amount of power on the output side must equal the amount put in originally, \( I_{\text{in}}V_{\text{in}} = I_{\text{out}}V_{\text{out}} \), so no matter what factor the voltage is reduced by, the current is increased by the same factor.

\[ g / A \text{ transformer.} \]

### Discussion Questions

◊ Suppose the bar magnet in figure f on page 687 has a magnetic field pattern that emerges from its top, circling around and coming back in the bottom. This field is created by electrons orbiting atoms inside the magnet. Are these atomic currents clockwise or counterclockwise as seen from above? In what direction is the current flowing in the circuit?

◊ We have a circling atomic current inside the circling current in the wires. When we have two circling currents like this, they will make torques on each other that will tend to align them in a certain way. Since currents in the same direction attract one another, which way is the torque made by the wires on the bar magnet? Verify that due to this torque, mechanical work has to be done in order to crank the generator.

Faraday's results leave us in the dark about several things:

- They don't explain why induction effects occur.
- The relationship \( \Gamma_E \propto -\partial B/\partial t \) tells us that a changing magnetic field creates an electric field in the surrounding region of space, but the phrase "surrounding region of space" is vague, and needs to be
made mathematical.

• Suppose that we can make the “surrounding region of space” idea more well defined. We would then want to know the proportionality constant that has been hidden by the \( \propto \) symbol. Although experiments like Faraday’s could be used to find a numerical value for this constant, we would like to know why it should have that particular value.

We can get some guidance from the example of a car’s alternator (which just means generator), referred to in the self-check on page 687. To keep things conceptually simple, I carefully avoided mentioning that in a real car’s alternator, it isn’t actually the permanent magnet that spins. The coil is what spins. The choice of design h/1 or h/2 is merely a matter of engineering convenience, not physics. All that matters is the relative motion of the two objects.

\[
\text{(1)}
\]

\[
\text{(2)}
\]

h / It doesn't matter whether it's the coil or the permanent magnet that spins. Either way, we get a functioning generator.

This is highly suggestive. As discussed at the beginning of this chapter, magnetism is a relativistic effect. From arguments about relative motion, we concluded that moving electric charges create magnetic fields. Now perhaps we can use reasoning with the same flavor to show that changing magnetic fields produce curly electric fields. Note that figure h/2 doesn't even require induction. The protons and electrons in the coil are moving through a magnetic field, so they experience forces. The protons can't flow, because the coil is a solid substance, but the electrons can, so a current is induced.\(^7\)

Now if we're convinced that figure h/2 produces a current in the coil, then it seems very plausible that the same will happen in figure h/1, which implies the existence of induction effects. But this example involves circular motion, so it doesn't quite work as a way of proving that induction exists. When we say that motion is relative, we only mean straight-line motion, not circular motion.

A more ironclad relativistic argument comes from the arrangement shown in figure i. This is also a generator --- one that is impractical, but much easier to understand.
Flea 1 doesn't believe in this modern foolishness about induction. She's sitting on the bar magnet, which to her is obviously at rest. As the square wire loop is dragged away from her and the magnet, its protons experience a force out of the page, because the cross product $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$ is out of the page. The electrons, which are negatively charged, feel a force into the page. The conduction electrons are free to move, but the protons aren't. In the front and back sides of the loop, this force is perpendicular to the wire. In the right and left sides, however, the electrons are free to respond to the force. Note that the magnetic field is weaker on the right side. It's as though we had two pumps in a loop of pipe, with the weaker pump trying to push in the opposite direction; the weaker pump loses the argument. We get a current that circulates around the loop. There is no induction going on in this frame of reference; the forces that cause the current are just the ordinary magnetic forces experienced by any charged particle moving through a magnetic field.

Flea 2 is sitting on the loop, which she considers to be at rest. In her frame of reference, it's the bar magnet that is moving. Like flea 1, she observes a current circulating around the loop, but unlike flea 1, she cannot use magnetic forces to explain this current. As far as she is concerned, the electrons were initially at rest. Magnetic forces are forces between moving charges and other moving charges, so a magnetic field can never accelerate a charged particle starting from rest. A force that accelerates a charge from rest can only be an electric force, so she is forced to conclude that there is an electric field in her region of space. This field drives electrons around and around in circles, so it is apparently violating the loop rule --- it is a curly field. What reason can flea 2 offer for the existence of this electric field pattern? Well, she's been noticing that the magnetic field in her region of space has been changing, possibly because that bar magnet over there has been getting farther away. She observes that a changing magnetic field creates a curly electric field.
We therefore conclude that induction effects must exist based on the fact that motion is relative. If we didn't want to admit induction effects, we would have to outlaw flea 2's frame of reference, but the whole idea of relative motion is that all frames of reference are created equal, and there is no way to determine which one is really at rest.

This whole line of reasoning was not available to Faraday and his contemporaries, since they thought the relative nature of motion only applied to matter, not to electric and magnetic fields. But with the advantage of modern hindsight, we can understand in fundamental terms the facts that Faraday had to take simply as mysterious experimental observations. For example, the geometric relationship shown in figure d follows directly from the direction of the current we deduced in the story of the two fleas.

We can also answer the other questions posed on page 688. The divide-and-conquer approach should be familiar by now. We first determine the circulation \( \Gamma_E \) in the case where the wire loop is very tiny, \( j \). Then we can break down any big loop into a grid of small ones; we've already seen that when we make this kind of grid, the circulations add together. Although we'll continue to talk about a physical loop of wire, as in figure i, the tiny loop can really be just like the edges of an Ampèrian surface: a mathematical construct that doesn't necessarily correspond to a real object.

\[
\begin{equation*}
B(x+dx)-B(x)=\frac{\partial B}{\partial x}\,dx
\end{equation*}
\]

for the difference in the strength of the field between the left and right sides. In the frame of reference where the loop is moving, a charge \( q \) moving along with the loop at velocity \( v \) will experience a magnetic force \( qvB\hat{y} \). In the frame moving along with the loop, this is interpreted as an electrical force, \( qE\hat{y} \). Observers in the two frames agree on how much force there is, so in the loop's frame, we have an electric field \( vB\hat{y} \). This field is perpendicular to the front and back sides of the loop, BC and DA, so

\( j \) / A new version of figure i with a tiny loop. The point of view is above the plane of the loop. In the frame of reference where the magnetic field is constant, the loop is moving to the right.
there is no contribution to the circulation along these sides, but there is a counterclockwise contribution to the circulation on CD, and smaller clockwise one on AB. The result is a circulation that is counterclockwise, and has an absolute value
\[
|\Gamma_E| = |E(x)dy - E(x+dx)dy| = |v [B(x)-B(x+dx)]|dy
\]
\[
= |v B_x|dy
\]
\[
= \left|v \frac{\partial B}{\partial x}\right|dx\,dy
\]
\[
= \left|\frac{dx}{dt} \frac{\partial B}{\partial x}\right|dxdy
\]
\[
= \left|\frac{\partial B}{\partial t}\right|dA.
\]

Using a right-hand rule, the counterclockwise circulation is represented by pointing one's thumb up, but the vector \(\frac{\partial \mathbf{B}}{\partial t}\) is down. This is just a rephrasing of the geometric relationship shown in figure d on page 686. We can represent the opposing directions using a minus sign,
\[
\Gamma_E = -\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A}.
\]

Although this derivation was carried out with everything aligned in a specific way along the coordinate axes, it turns out that this relationship can be generalized as a vector dot product,
\[
\Gamma_E = -\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{A}.
\]

Finally, we can take a finite-sized loop and break down the circulation around its edges into a grid of tiny loops. The circulations add, so we have
\[
\Gamma_E = -\sum \frac{\partial \mathbf{B}_j}{\partial t} \cdot d\mathbf{A}_j.
\]

This is known as Faraday's law. (I don't recommend memorizing all these names.) Mathematically, Faraday's law is very similar to the structure of Ampère's law: the circulation of a field around the edges of a surface is equal to the sum of something that points through the

If the loop itself isn't moving, twisting, or changing shape, then the area vectors don't change over time, and we can move the derivative outside the sum, and rewrite Faraday's law in a slightly more transparent form:
\[
\Gamma_E = -\frac{\partial}{\partial t} \sum \mathbf{B}_j \cdot d\mathbf{A}_j.
\]

A changing magnetic flux makes a curly electric field. You might think based on Gauss' law for magnetic fields that \(\Phi_B\) would be identically zero. However, Gauss' law only applies to surfaces that are closed, i.e., have no edges.

**self-check:**

Check that the units in Faraday's law work out. An easy way to approach this is to use the fact that \(vB\) has the same units as \(E\), which can be seen by comparing the equations for magnetic and electric forces used above.

(answer in the back of the PDF version of the book)

**Example 20: A pathetic generator**

\(\triangleleft\) The horizontal component of the earth's magnetic field varies from zero, at a magnetic pole, to about 
\(10^{-4}\) T near the equator. Since the distance from the equator to a pole is about \(10^7\) m, we can estimate, very
roughly, that the horizontal component of the Earth's magnetic field typically varies by about \(10^{\text{-11}}\) T/m as you go north or south. Suppose you connect the terminals of a one-ohm lightbulb to each other with a loop of wire having an area of 1 \(\text{m}^2\). Holding the loop so that it lies in the east-west-up-down plane, you run straight north at a speed of 10 m/s, how much current will flow? Next, repeat the same calculation for the surface of a neutron star. The magnetic field on a neutron star is typically \(10^9\) T, and the radius of an average neutron star is about \(10^4\) m.

Let's work in the frame of reference of the running person. In this frame of reference, the Earth is moving, and therefore the local magnetic field is changing in strength by \(10^{\text{-9}}\) T/s. This rate of change is almost exactly the same throughout the interior of the loop, so we can dispense with the summation, and simply write Faraday's law as

\[
\Gamma_E = -\frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{A}.
\]

Since what we estimated was the rate of change of the horizontal component, and the vector \(\mathbf{A}\) is horizontal (perpendicular to the loop), we can find this dot product simply by multiplying the two numbers:

\[
\Gamma_E = (10^{\text{-9}}\text{T/s})(1\text{m}^2) = 10^{\text{-9}}\text{T}\cdot\text{m}^2/\text{s} = 10^{\text{-9}}\text{V}.
\]

This is certainly not enough to light the bulb, and would not even be easy to measure using the most sensitive laboratory instruments.

Now what about the neutron star? We'll pretend you're tough enough that its gravity doesn't instantly crush you. The spatial variation of the magnetic field is on the order of \((10^9/10^4)=10^5\) T/m. If you can run north at the same speed of 10 m/s, then in your frame of reference there is a temporal (time) variation of about \(10^6\) T/s, and a calculation similar to the previous one results in an emf of \(10^6\) V! This isn't just strong enough to light the bulb, it's sufficient to evaporate it, and kill you as well!

It might seem as though having access to a region of rapidly changing magnetic field would therefore give us an infinite supply of free energy. However, the energy that lights the bulb is actually coming from the mechanical work you do by running through the field. A tremendous force would be required to make the wire loop move through the neutron star's field at any significant speed.

**Example 21: Speed and power in a generator**

Figure k shows three graphs of the magnetic flux through a generator's coils as a function of time. In graph 2, the generator is being cranked at twice the frequency. In 3, a permanent magnet with double the strength has been used. In 4, the generator is being cranked in the opposite direction. Compare the power generated in figures 2-4 with the the original case, 1.
If the flux varies as \(\Phi = A \sin \omega t\), then the time derivative occurring in Faraday's law is \(\frac{\partial \Phi}{\partial t} = A \omega \cos \omega t\). The absolute value of this is the same as the absolute value of the emf, \(\Gamma_E\). The current through the lightbulb is proportional to this emf, and the power dissipated depends on the square of the current \((P = I^2R)\), so \((P \propto A^2 \omega^2)\). Figures 2 and 3 both give four times the output power (and require four times the input power). Figure 4 gives the same result as figure 1; we can think of this as a negative amplitude, which gives the same result when squared.

**Example 22: An approximate loop rule**

Figure 1/1 shows a simple RL circuit of the type discussed in the last chapter. A current has already been established in the coil, let's say by a battery. The battery was then unclipped from the coil, and we now see the circuit as the magnetic field in and around the inductor is beginning to collapse. I've already cautioned you that the loop rule doesn't apply in nonstatic situations, so we can't assume that the readings on the four voltmeters add up to zero. The interesting thing is that although they don't add up to exactly zero in this circuit, they very nearly do. Why is the loop rule even approximately valid in this situation?
The reason is that the voltmeters are measuring the emf $\Gamma_E$ around the path shown in figure 1/2, and the stray field of the solenoid is extremely weak out there. In the region where the meters are, the arrows representing the magnetic field would be too small to allow me to draw them to scale, so I have simply omitted them. Since the field is so weak in this region, the flux through the loop is nearly zero, and the rate of change of the flux, $\partial\Phi_B/\partial t$, is also nearly zero. By Faraday's law, then, the emf around this loop is nearly zero.

Now consider figure 1/3. The flux through the interior of this path is not zero, because the strong part of the field passes through it, and not just once but many times. To visualize this, imagine that we make a wire frame in this shape, dip it in a tank of soapy water, and pull it out, so that there is a soap-bubble film spanning its interior. Faraday's law refers to the rate of change of the flux through a surface such as this one. (The soap film tends to assume a certain special shape which results in the minimum possible surface area, but Faraday's law would be true for any surface that filled in the loop.) In the coiled part of the wire, the soap makes a three-dimensional screw shape, like the shape you would get if you took the steps of a spiral staircase and smoothed them into a ramp. The loop rule is going to be strongly violated for this path.

We can interpret this as follows. Since the wire in the solenoid has a very low resistance compared to the resistances of the light bulbs, we can expect that the electric field along the corkscrew part of loop 1/3 will be very small. As an electron passes through the coil, the work done on it is therefore essentially zero, and the true emf along the coil is zero. In figure 1/1, the meter on top is therefore not telling us the actual emf experienced by an electron that passes through the coil. It is telling us the emf experienced by an electron that passes through the meter itself, which is a different quantity entirely. The other three meters, however, really do tell us the emf through the bulbs, since there are no magnetic fields where they are, and therefore no funny induction effects.

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